Math 220 - Practice Final (Fall 2004) Solutions

1. (a) $f'(x) = -\frac{1}{x^2} - 2\sin(x) + 3\sec^2(x) - 4\csc^2(x) + \frac{5}{x} + 6e^x + \frac{1}{\ln(7)x} + \ln(8)8^x + \frac{9}{1+x^2} + \frac{1}{\sqrt{1-x^2}}$.

   (b) We have

$$
\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h)-f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2-4}{h} \\
&= \lim_{h\to 0} \frac{4+4h+h^2-4}{h} \\
&= \lim_{h\to 0} 4+h \\
&= 4.
\end{aligned}
$$

   (c) This is the area of a rectangle of width 3 and height 1, plus a triangle of width 3 and height 6. So the area is $3\cdot 1 + \frac{1}{2}3\cdot 6 = 12$.

4. (a) $f'(x) = (2x+1)(x^3-3x^2+x+1) + (x^2+x+1)(3x^2-6x+1)$.

   (b) $g'(x) = \frac{1\cdot(x^2+1)-(x+1)(2x)}{(x^2+1)^2}$.

   (c) $p'(x) = 10(1+x^4)^9 \cdot (4x^3)$.

   (d) $q'(x) = \cos\left(\frac{1}{xe^{2x}}\right)\cdot\left(-\frac{1}{x^2e^{4x}}\right)\cdot\left(e^{2x}+2xe^{2x}\right) = -\frac{1+2x}{x^2e^{2x}}\cos\left(\frac{1}{xe^{2x}}\right)$.

5. (a) Either by recognizing this as a derivative or by a calculation, the answer is $2x$.

   (b) Since numerator and denominator both tend to 0, L'Hospital's rule applies. Differentiating numerator and denominator, we get

$$
\begin{aligned}
\lim_{x\to 0} \frac{1-e^x}{\sin(x)} &= \lim_{x\to 0} \frac{-e^x}{\cos(x)} \\
&= 1.
\end{aligned}
$$

   (c) For $x > 1$, we have $\frac{x-1}{|x-1|} = 1$, while for $x < 1$ we have $\frac{x-1}{|x-1|} = -1$. Therefore the limit does not exist.

   (d) Putting the difference over a common denominator, $\frac{1+x}{x\cos(x)} - \frac{1}{x} = \frac{1+x-\cos(x)}{x\cos(x)}$. This is of indeterminate form $0/0$, so L'Hospital's rule applies. Differentiating numerator and denominator, we get $\frac{1+\sin(x)}{\cos(x)-x\sin(x)}$, which tends to 1 as $x \to 0$. Thus

$$
\lim_{x\to 0}\left(\frac{1+x}{x\cos(x)} - \frac{1}{x}\right) = 1.
$$

6. (a)

$$\begin{aligned}
R_4 &= \sum_{i=1}^{4} f(c_i)(x_i - x_{i-1}) \\
&= \sum_{i=1}^{4} c_i^2(2i - 2(i-1)) \\
&= 2\sum_{i=1}^{4} c_i^2 \\
&= 2(1 + 9 + 25 + 49) \\
&= 168.
\end{aligned}$$

(b) $\int_0^8 x^2\,dx = [\frac{x^3}{3}]_0^8 = 170\frac{2}{3}$.

(c) $\int \left(x^2 + \frac{2}{x} + 3\cos(x) + \frac{4}{\sqrt{1-x^2}} + \frac{5}{1+x^2}\right) dx = \frac{x^3}{3} + 2\ln(x) + 3\sin(x) + 4\sin^{-1}(x) + 5\tan^{-1}(x) + C.$

(d) By the fundamental theorem of calculus, $F'(x) = x^2 e^{x^2}$.

7. (a) Let $L(x)$ be the linear approximation near $a = 1000$:

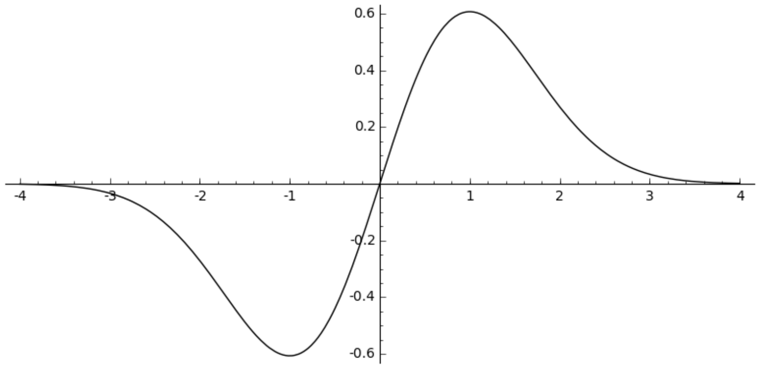
$$\begin{aligned}
L(x) - f(a) &= f'(a)(x - a) \\
&= \frac{1}{3}1000^{-2/3}(x - 1000) \\
&= \frac{1}{300}(x - 1000) \\
\sqrt[3]{1003} - \sqrt[3]{1000} &\approx L(1003) - f(1000) \\
&= \frac{1}{300}1003 - 1000 \\
&= 0.01.
\end{aligned}$$

8. (a) $f(x)$ is increasing when $f'(x) > 0$, which occurs when $-1 < x < 1$. It is decreasing when $f'(x) < 0$, which occurs when $x < -1$ or $x > 1$. The only local minimum is therefore at $x = -1$, where $f(x) = -1$ and the only local maximum is at $x = 1$, where $f(x) = 1$. Here we use the first derivative test to determine whether each point is a minimum or maximum, and we will see in part (c) that these are also global extreme values.

(b) $f(x)$ is concave up when $f''(x) > 0$, which occurs when $x > \sqrt{3}$ or $-\sqrt{3} < x < 0$. Similarly, $f(x)$ is concave down when $f''(x) < 0$, which occurs when $x < -\sqrt{3}$ or $0 < x < \sqrt{3}$.

(c) As $x \to \pm\infty$, the exponent $(1 - x^2)/2 \to -\infty$ and thus $f(x) \to 0$ (either using L'Hospital's rule or the fact that exponentials dominate polynomials). Therefore $y = 0$ is a horizontal asymptote.

(d) Here's the graph for comparison with your sketch.



9. If x is the width/length of the box and h is its height, then the cost is $4x^2 + 8xh$ and the volume is $x^2h = 1000$. Solving for h and substituting, we seek to minimize the function

$$4x^2 + \frac{8000}{x}.$$

Differentiating and setting equal to zero, we get

$$8x - \frac{8000}{x^2} = 0,$$

so $x = 10$. Thus the dimensions are $10 \times 10 \times 10$.

10. When the water has height h, the volume is $20000h$, so

$$\frac{dV}{dt} = 20000\frac{dh}{dt}.$$

Since $\frac{dV}{dt} = 2\text{m}^3/\text{min}$, the depth of the water is decreasing at a rate of $\frac{1}{10000}$ meter per minute, or 0.1 millimeters per minute.

11. (a) Taking logarithms, we get $\ln(q) = \ln(x^x) = x\ln(x)$. Differentiating, we have $\frac{q'}{q} = \ln(x) + 1$. Therefore $q'(x) = (\ln(x) + 1)x^x$.

(b) Differentiating, we have

$$3y^2y' + y + xy' = 0.$$

Solving for y' gives

$$y'(x) = -\frac{y}{3y^2 + x}.$$