Math 220 (6pm section) - Exam 2 Solutions

1. Determine the derivatives of the following functions. (5 points each)

   (a) $f(x) = \tan^{-1}(e^x)$

   Solution. Using the chain rule,

$$f'(x) = \frac{1}{1 + (e^x)^2} \cdot e^x = \frac{e^x}{1 + e^{2x}}$$

   (b) $\ln(x)^{\ln(x)}$

   Solution. Since $\ln(x)^{\ln(x)} = e^{\ln(x)\ln(\ln(x))}$, we can use the chain rule to get

$$\begin{aligned} f'(x) &= e^{\ln(x)\ln(\ln(x))}\left(\frac{1}{x} \cdot \ln(\ln(x)) + \ln(x)\frac{1}{\ln(x)}\frac{1}{x}\right) \\ &= \frac{1}{x}\ln(x)^{\ln(x)}\left(\ln(\ln(x)) + 1\right) \end{aligned}$$

   (c) $e^{e^{e^x}}$

   Solution. Using the chain rule twice,

$$f'(x) = e^{e^{e^x}} \cdot e^{e^x} \cdot e^x.$$

   (d) $x\sinh(\ln(x))$

   Solution. Using the product rule and chain rule,

$$\begin{aligned} f'(x) &= \sinh(\ln(x)) + x\cosh(\ln(x)) \cdot \frac{1}{x} \\ &= \sinh(\ln(x)) + \cosh(\ln(x)). \end{aligned}$$

   Alternatively, note that $x\sinh(\ln(x)) = x\frac{e^{\ln(x)} - e^{-\ln(x)}}{2} = \frac{x^2-1}{2}$, so the derivative is just x. These two answers are the same, since

$$\begin{aligned} \sinh(\ln(x)) + \cosh(\ln(x)) &= \frac{e^{\ln(x)} - e^{-\ln(x)}}{2} + \frac{e^{\ln(x)} + e^{-\ln(x)}}{2} \\ &= \frac{x - 1/x}{2} + \frac{x + 1/x}{2} \\ &= x. \end{aligned}$$

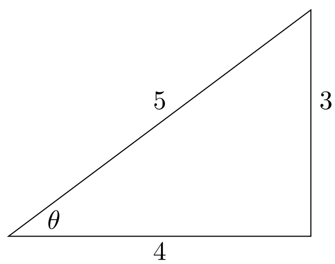
2. Evaluate so that your answer is a fraction. (5 points each)

   (a) $\ln(\cosh(2) - \sinh(2)) =$

Solution. We have

$$\begin{aligned}\ln(\cosh(2) - \sinh(2)) &= \ln\left(\frac{e^2 + e^{-2}}{2} - \frac{e^2 - e^{-2}}{2}\right)\\ &= \ln\left(\frac{2e^{-2}}{2}\right)\\ &= -2.\end{aligned}$$

(b) $\cot(\cos^{-1}(\frac{4}{5})) =$



Solution.

If $\theta = \cos^{-1}(\frac{4}{5})$, then it is the measure of the marked angle above. Since the cotangent is the ratio of adjacent divided by opposite, we get

$$\cot(\cos^{-1}(\frac{4}{5})) = \frac{4}{3}.$$

3. Determine each limit. Show your work. (6 points each)

(a) $\lim_{x\to 0^+} x\ln(x)$

Solution. Rewrite $x\ln(x) = \frac{\ln(x)}{1/x}$. Evaluating at $x = 0^+$ gives the indeterminate form $\frac{-\infty}{\infty}$, so L'Hospital's rule applies. Differentiating top and bottom, we consider

$$\lim_{x\to 0^+} \frac{1/x}{-1/x^2} = \lim_{x\to 0^+} -x = 0.$$

Therefore the original limit is also 0.

(b) $\lim_{x\to 0} \cosh(x)^{1/x^2}$

Solution. Evaluating at $x = 0$ yields the indeterminate form 1^∞, which leads us to take the natural logarithm of the expression and try to compute

$$\lim_{x\to 0} \ln\left(\cosh(x)^{1/x^2}\right) = \lim_{x\to 0} \frac{\ln(\cosh(x))}{x^2}.$$

This is of form $\frac{0}{0}$, so L'Hospital's rule applies. Differentiating top and bottom, we are led to consider

$$\lim_{x\to 0} \frac{\sinh(x)/\cosh(x)}{2x} = \lim_{x\to 0} \frac{\tanh(x)}{2x}.$$

Again, this is of form $\frac{0}{0}$, so we apply L'Hospital's rule again and consider

$$\lim_{x\to 0} \frac{\operatorname{sech}^2(x)}{2} = \frac{1}{2}.$$

Therefore

$$\lim_{x\to 0} \ln\left(\cosh(x)^{1/x^2}\right) = \frac{1}{2}$$

and

$$\lim_{x\to 0} \cosh(x)^{1/x^2} = e^{1/2} = \sqrt{e}.$$

4. Find the point on the line $y = 2x - 5$ closest to the origin. (12 points)

Solution. Suppose (x, y) is a point on the line. We seek to minimize the squared distance to the origin is $D = x^2 + y^2$, and we have $y = 2x - 5$ since (x, y) is on the line. Substituting, we find

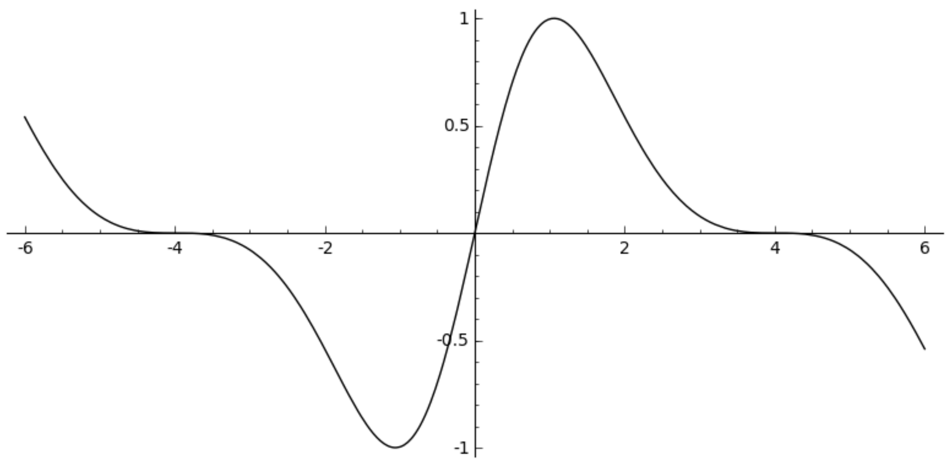
$$D = x^2 + (2x - 5)^2,$$

and differentiating,

$$D' = 2x + 2 \cdot (2x - 5) \cdot 2 = 10x - 20.$$

Setting $D' = 0$ we get $x = 2$ and thus $y = 2 \cdot 2 - 5 = -1$. We know this is a minimum either from geometric reasoning (a minimum value clearly exists and this is the only critical point), the first derivative test ($D' < 0$ when $x < 2$ and $D' > 0$ when $x > 2$) or the second derivative test ($D'' = 10 > 0$). So the closest point to the origin is $(2, -1)$.

5. Shown below is the graph of the derivative $f'(x)$ of a function $f(x)$ ($f(x)$ is NOT shown).



Within the interval shown, answer the following questions about $f(x)$ (NOT $f'(x)$). Briefly explain your reasoning, but feel free to round numbers to the nearest integer. (2 points each)

(a) Where is $f(x)$ increasing?

Solution. $f(x)$ is increasing on $(-6, -4)$ and $(0, 4)$ since this is where $f'(x) > 0$.

(b) Where is $f(x)$ decreasing?

Solution. $f(x)$ is increasing on $(-4, 0)$ and $(4, 6)$ since this is where $f'(x) < 0$.

(c) What are the local maxima of $f(x)$, and how do you know they are maxima?

Solution. The local maxima are at $x = -4$ and $x = 4$ since these are the points where $f'(x) = 0$ and $f'(x)$ is decreasing.

(d) What are the local minima of $f(x)$, and how do you know they are minima?

Solution. The only local minimum is at $x = 0$ since this is the point where $f'(x) = 0$ and $f'(x)$ is increasing.

(e) Where is $f(x)$ concave up?

Solution. $f(x)$ is concave up on $(-1, 1)$ since this is where $f'(x)$ is increasing.

(f) Where is $f(x)$ concave down?

Solution. $f(x)$ is concave down on $(-6, -1)$ and $(1, 6)$ since this is where $f'(x)$ is decreasing.

(g) Where are the inflection points of $f(x)$?

Solution. The inflection points of $f(x)$ are at -1 and 1 since these are where $f'(x)$ changes from increasing to decreasing or vice versa.

6. Let $f(x) = x^{2/3}(x^2 - 16)$. Find the minimum and maximum values of $f(x)$ on the interval $[-3, 3]$. Show your work. (12 points)

Solution. Expanding, we have $f(x) = x^{8/3} - 16x^{2/3}$, so

$$f'(x) = \frac{8}{3}x^{5/3} - \frac{32}{3}x^{-1/3} = \frac{8}{3}x^{-1/3}(x^2 - 4).$$

Therefore the critical points are at $x = 0$ (where $f'(x)$ is undefined) and $x = \pm 2$ (where $'f(x) = 0$). To find the maxima and minima of $f(x)$ on $[-3, 3]$ we must consider these critical points and the endpoints $x = \pm 3$. Note that $f(x)$ is even, so $f(-2) = f(2)$ and $f(-3) = f(3)$. We compute

$$\begin{aligned} f(0) &= 0^{2/3}(0^2 - 16) = 0 \\ f(2) &= 2^{2/3}(2^2 - 16) = -12\sqrt[3]{4} \\ f(3) &= 3^{2/3}(3^2 - 16) = -7\sqrt[3]{9} \end{aligned}$$

The maximum value is thus 0, since the other two possibilities are negative. By the first derivative test, $f(x)$ has a local minimum at $x = 2$, so $f(2) < f(3)$. Alternatively, you can compare $f(2)^3 = -1728 \cdot 4$ to $f(3)^3 = -343 \cdot 9$ to see that $f(2)$ is smaller.

The final result is that the maximum value of $f(x)$ on $[-3, 3]$ is 0 and the minimum is $-12\sqrt[3]{4}$.

7. A sample of plutonium initially has a mass of $128g$, but after 30 years there is only $32g$ remaining. How much will be left after 75 years? Show your work. (8 points)

Solution. We use the model $m(t) = m_0 e^{kt}$ for radioactive decay. Evaluating at $t = 0$ gives $m(0) = m_0 = 128$ and at $t = 30$ gives

$$\begin{aligned} 128e^{30k} &= 32 \\ e^{30k} &= \frac{1}{4} \\ 30k &= \ln(1/4) \\ k &= \ln(1/4)/30 \end{aligned}$$

So

$$\begin{aligned} m(75) &= 128e^{\ln(1/4) \cdot 75/30} \\ &= 128(1/4)^{5/2} \\ &= 128(1/2)^5 \\ &= 4. \end{aligned}$$

There are $4g$ of plutonium after 75 years.

8. Suppose that the functions $f(x)$ and $g(x)$ are differentiable, with values given in the following table.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	-1	3	2	-2
1	0	2	0	-3
2	1	1/2	-1	-1/2
3	3	1	-2	-1

Suppose that $h(x) = g(f(x))$. What is $(h^{-1})'(0)$? Show your work. (12 points)

Solution. We have

$$(h^{-1})'(x) = \frac{1}{h'(h^{-1}(x))}$$

(if you forget this formula, you can derive it by differentiating the identity $h(h^{-1}(x)) = x$).

To compute $h^{-1}(0)$, we look for an a with $g(f(a)) = 0$. We break this task up into two steps: find b so that $g(b) = 0$ and then find a so that $f(a) = b$. Examining the table, the only input with $g(b) = 0$ is $b = 1$, and the only input with $f(a) = 1$ is $a = 2$. So $h^{-1}(0) = 2$.

Now we use the chain rule to compute the derivative of h in terms of the derivatives of f and g:

$$\begin{aligned}
(h^{-1})'(0) &= \frac{1}{h'(h^{-1}(0))} \\
&= \frac{1}{h'(2)} \\
&= \frac{1}{g'(f(2)) \cdot f'(2)} \\
&= \frac{1}{g'(1) \cdot f'(2)} \\
&= \frac{1}{-3 \cdot 1/2} \\
&= -\frac{2}{3}.
\end{aligned}$$