# Math 220 - Practice Exam 1 (version B) Solutions

1. Give a value for each of the following limits. (4 points each)

   (a) $\lim_{x\to 1} \dfrac{\sqrt{x^2+3}-\sqrt{3x^2+1}}{x-1}$

   **Solution.** Multiply by the conjugate:

$$\begin{aligned}
\lim_{x\to 1} \frac{\sqrt{x^2+3}-\sqrt{3x^2+1}}{x-1} &= \lim_{x\to 1} \frac{\sqrt{x^2+3}-\sqrt{3x^2+1}}{x-1} \cdot \frac{\sqrt{x^2+3}+\sqrt{3x^2+1}}{\sqrt{x^2+3}+\sqrt{3x^2+1}} \\
&= \lim_{x\to 1} \frac{(x^2+3)-(3x^2+1)}{(x-1)(\sqrt{x^2+3}+\sqrt{3x^2+1})} \\
&= \lim_{x\to 1} \frac{-2x^2+2}{(x-1)(\sqrt{x^2+3}+\sqrt{3x^2+1})} \\
&= \lim_{x\to 1} \frac{-2(x-1)(x+1)}{(x-1)(\sqrt{x^2+3}+\sqrt{3x^2+1})} \\
&= \lim_{x\to 1} \frac{-2(x+1)}{\sqrt{x^2+3}+\sqrt{3x^2+1}} \\
&= \frac{-2(1+1)}{\sqrt{1+3}+\sqrt{3+1}} \\
&= \frac{-4}{4} \\
&= -1
\end{aligned}$$

   (b) $\lim_{x\to 2^+} \dfrac{|1-x^2|}{x-2}$

   **Solution.** Since $\lim_{x\to 2^+}|1-x^2| = 3$ and $\lim_{x\to 2^+}(x-2) = 0$, the function $\frac{|1-x^2|}{x-2}$ has a vertical asymptote at $x = 2$. Near $x = 2$, $1-x^2 < 0$ but $|1-x^2| = x^2-1 > 0$. Therefore

$$\lim_{x\to 2^+} \frac{|1-x^2|}{x-2} = +\infty.$$

   (c) $\lim_{t\to 0} \left(\dfrac{1}{t} - \dfrac{1}{t^2-t}\right)$

   **Solution.** We combine into one fraction:

$$\begin{aligned}
\frac{1}{t} - \frac{1}{t^2-t} &= \frac{t^2-t}{t(t^2-t)} - \frac{t}{t(t^2-t)} \\
&= \frac{t^2-2t}{t(t^2-t)} \\
&= \frac{t-2}{t^2-t}.
\end{aligned}$$

The numerator is negative near $t = 0$; the denominator is negative for $0 < t < 1$ and positive for $t < 0$, so

$$\lim_{t \to 0^+} \frac{t-2}{t^2 - t} = +\infty$$
$$\lim_{t \to 0^-} \frac{t-2}{t^2 - t} = -\infty.$$

Since the two one-sided limits are different, $\lim_{t \to 0} \left( \frac{1}{t} - \frac{1}{t^2 - t} \right)$ does not exist.

2. Determine the derivatives of the following functions. (4 points each)

(a) $f(x) = \sqrt[3]{x} + (1+x)^{99}$

**Solution.** By the power rule and chain rule,

$$f'(x) = \frac{1}{3} x^{-2/3} + 99(1+x)^{98}.$$

(b) $f(x) = (x^3 + 1)^6 \sin(x)$

**Solution.** By the product rule and chain rule,

$$\begin{aligned} f'(x) &= 6(x^3+1)^5(3x^2)\sin(x) + (x^3+1)^6\cos(x) \\ &= (x^3+1)^5 \left(18x^2\sin(x) + (x^3+1)\cos(x)\right). \end{aligned}$$

(c) $f(x) = \dfrac{x^3 + x}{3x^2 - 1}$

**Solution.** By the quotient rule,

$$f'(x) = \frac{(3x^2+1)(3x^2-1) - (x^3+x)(6x)}{(3x^2-1)^2}.$$

(d) $f(x) = \tan(\cos(x^2))$

**Solution.** By the chain rule,

$$f'(x) = \sec^2(\cos(x^2)) \cdot (-\sin(x^2)) \cdot (2x).$$

(e) $f(x) = \dfrac{1}{x + \sin^2(x + x^2)}$

**Solution.** By the chain rule, using the fact that $f(x) = (x + \sin^2(x+x^2))^{-1}$, we have

$$f'(x) = \frac{-\left(1 + 2\sin(x+x^2)\cos(x+x^2)(1+2x)\right)}{(x + \sin^2(x+x^2))^2}.$$

3. Suppose that $\lim_{x \to 1} \dfrac{f(x) - 4}{x - 1} = 9$. Find $\lim_{x \to 1} f(x)$. Justify your answer. (8 points)

**Solution.** Since the limit of the denominator is zero, in order for the whole limit to have a finite value, the limit of the numerator must be zero as well. This implies that $\lim_{x \to 1} f(x) = 4$.

Alternatively, we can solve for $\lim_{x \to 1} f(x)$:

$$\begin{aligned} \lim_{x \to 1} \frac{f(x)-4}{x-1} &= 9 \\ \lim_{x \to 1} f(x) - 4 &= 9 \lim_{x \to 1} (x-1) \\ \lim_{x \to 1} f(x) &= 4 + 9 \lim_{x \to 1} (x - 1) \\ &= 4 + 9 \cdot 0 \\ &= 4. \end{aligned}$$

4. Determine the equation of the tangent line to the curve

$$x\sin(y) - x^2\cos(y) = 1$$

at the point $(1, \pi/2)$. (10 points)

**Solution.** Implicitly differentiate with respect to $x$:

$$\sin(y) + x\cos(y)y' - 2x\cos(y) - x^2(-\sin(y))y' = 0.$$

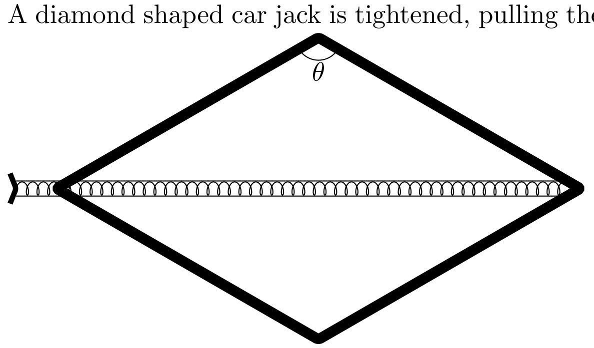
Substitute $x = 1$ and $y = \pi/2$:

$$\begin{aligned} 1 + 1 \cdot 0 \cdot y' - 2 \cdot 1 \cdot 0 - 1^2 \cdot (-1) \cdot y' &= 0 \\ 1 + y' &= 0 \\ y' &= -1. \end{aligned}$$

Using point-slope form, the equation of the tangent line is

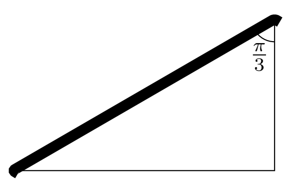
$$y - \pi/2 = -(x-1).$$

5. A diamond shaped car jack is tightened, pulling the left and right corners together at a rate of 1mm/s.




Suppose that all sides of the jack are 300mm long. Find the rate at which the car is raised when $\theta = 2\pi/3$. Feel free to leave square roots in your answer. (10 points)

**Solution.** We consider the right triangle




It has hypotenuse $L = 300$, and therefore base $b = 300\sin(\pi/3) = 150\sqrt{3}$ and height $h = 300\cos(\pi/3) = 150$. We have

$$b^2 + h^2 = L^2,$$

which we may differentiate to get

$$2bb' + 2hh' = 2LL' = 0.$$

Since the corners are pulled together at 1mm/s, and this triangle is half the width of the jack overall, $b' = -1/2$. Thus

$$\begin{aligned} h' &= \frac{-bb'}{h} \\ &= \frac{(-150\sqrt{3})(-1/2)}{150} \\ &= \frac{\sqrt{3}}{2}. \end{aligned}$$

Since there are two such triangles vertically in the jack, the car is raised at a rate of $2h' = \sqrt{3}$mm/s.

I did not penalize mistakes regarding the factors of 2 that arose when splitting the diamond up into triangles.

6. Determine where the function $f(x) = \frac{x^2-x}{2x^2-1}$ has a horizontal tangent line. (8 points)

**Solution.** We set $f'(x) = 0$ and solve for $x$:

$$\begin{aligned} f'(x) &= \frac{(2x-1)(2x^2-1)-(x^2-x)(4x)}{(2x^2-1)^2} \\ &= \frac{4x^3-2x^2-2x+1-4x^3+4x^2}{(2x^2-1)^2} \\ &= \frac{2x^2-2x+1}{(2x^2-1)^2} \\ &= 0. \end{aligned}$$

This rational function will vanish exactly when the numerator does, so we must solve $2x^2-2x+1 = 0$. The discriminant of $2x^2-2x+1$ is $(-2)^2 - 4\cdot 2\cdot 1 = -4 < 0$, and thus this quadratic has no real roots (this manifests in the quadratic formula as square roots of $-4$). Therefore $f(x)$ has *no* horizontal tangent lines.

7. Suppose that $f(x)$ is a differentiable function with $f(1) = 8$ and $f'(1) = -3$. Let $h(x) = \sqrt{1+f(x^2)}$. Find $h'(1)$. (10 points)

**Solution.** We use the chain rule, then evaluate at $x = 1$.

$$\begin{aligned} h'(x) &= \frac{f'(x^2)\cdot(2x)}{2\sqrt{1+f(x^2)}} \\ &= \frac{xf'(x^2)}{\sqrt{1+f(x^2)}} \\ h'(1) &= \frac{1\cdot f'(1^2)}{\sqrt{1+f(1^2)}} \\ &= \frac{f'(1)}{\sqrt{1+f(1)}} \\ &= \frac{-3}{\sqrt{1+8}} \\ &= -1. \end{aligned}$$

8. Let $f(x) = \sqrt[3]{x}$.

(a) Find a linear approximation to $f(x)$ near $x = a$. (5 points)

**Solution.** Since $f(x) = x^{1/3}$, we have $f'(x) = \frac{1}{3}x^{-2/3}$. Then

$$\begin{aligned} f(x) &\approx f(a) + f'(a)(x-a) \\ &= \sqrt[3]{a} + \frac{1}{3\sqrt[3]{a^2}}(x-a). \end{aligned}$$

(b) Approximate $\sqrt[3]{8.012}$. (5 points)

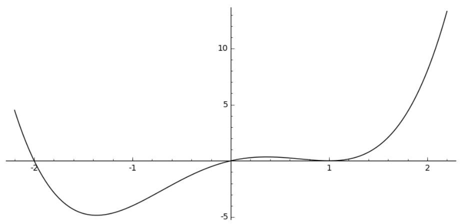
**Solution.** We set $a = 8$, which has cube root 2, and $x = 8.012$. Then

$$\begin{aligned}\sqrt[3]{8.012} &\approx \sqrt[3]{8} + \frac{1}{3\sqrt[3]{8^2}}(8.012 - 8)\\ &= 2 + \frac{1}{3 \cdot 4}(0.012)\\ &= 2.001.\end{aligned}$$

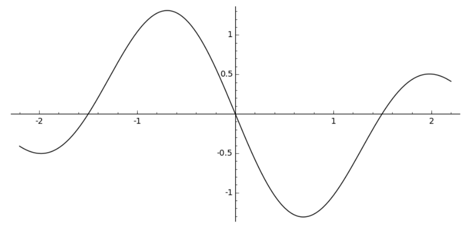
9. Match each graph with its derivative. (3 points per correct match)

$f(x)$ $f'(x)$

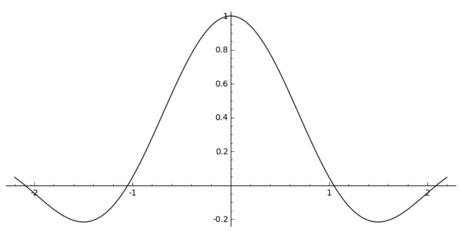
(a)



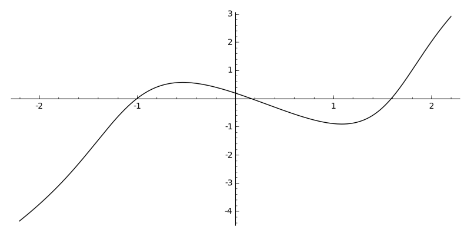

(1)



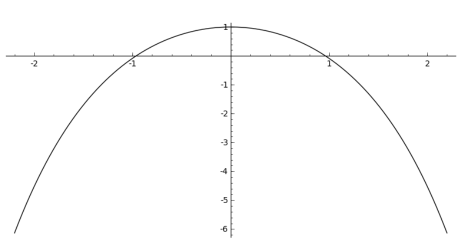

(b)



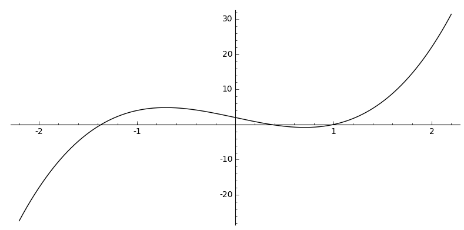

(2)



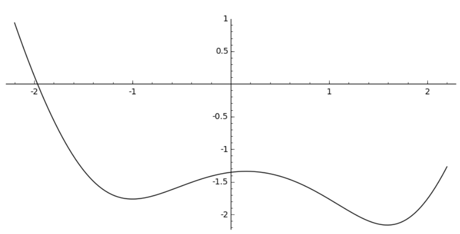

(c)




(3)




(d)

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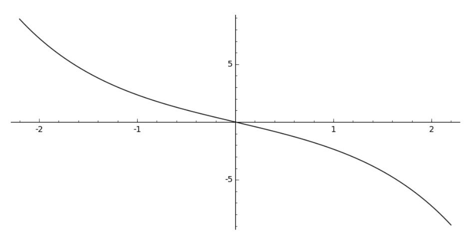


**Solution.**

(a) 3
(b) 1
(c) 4
(d) 2