

Math 220 - Practice Exam 1 (version A) Solutions

1. Give the limit if it exists

(a) $\lim_{x \rightarrow 2^-} \frac{|x^2 - 2x|}{x - 2}$

Solution. For $0 < x < 2$, $x^2 - 2x < 0$, so $\frac{|x^2 - 2x|}{x - 2} = \frac{2x - x^2}{x - 2} = \frac{x(2 - x)}{x - 2} = -x$. Therefore

$$\lim_{x \rightarrow 2^-} \frac{|x^2 - 2x|}{x - 2} = \lim_{x \rightarrow 2^-} (-x) = -2.$$

(b) $\lim_{x \rightarrow 2^+} \frac{|x^2 - 2x|}{x - 2}$

Solution. For $x > 2$, $x^2 - 2x > 0$, so $\frac{|x^2 - 2x|}{x - 2} = \frac{x^2 - 2x}{x - 2} = \frac{x(x - 2)}{x - 2} = x$. Therefore

$$\lim_{x \rightarrow 2^+} \frac{|x^2 - 2x|}{x - 2} = \lim_{x \rightarrow 2^+} x = 2.$$

(c) $\lim_{x \rightarrow 2} \frac{\sqrt{x+2} - 2}{x - 2}$

Solution. Multiply numerator and denominator by the conjugate, then cancel terms.

$$\begin{aligned} \lim_{x \rightarrow 2} \frac{\sqrt{x+2} - 2}{x - 2} &= \lim_{x \rightarrow 2} \frac{(\sqrt{x+2} - 2)(\sqrt{x+2} + 2)}{(x - 2)(\sqrt{x+2} + 2)} \\ &= \lim_{x \rightarrow 2} \frac{x + 2 - 4}{(x - 2)(\sqrt{x+2} + 2)} \\ &= \lim_{x \rightarrow 2} \frac{1}{\sqrt{x+2} + 2} \\ &= \frac{1}{\sqrt{2+2} + 2} \\ &= \frac{1}{4} \end{aligned}$$

2. Determine the derivatives of the following functions:

(a) $f(x) = x^4 - 2\sqrt{x} + 1 - \frac{2}{x^3} + \pi x^\pi$,

Solution.

$$f'(x) = 4x^3 - \frac{1}{\sqrt{x}} + \frac{6}{x^4} + \pi^2 x^{\pi-1}.$$

(b) $f(x) = (x^2 - 4x + 5)^6 (2x^3 - 1)^3$,

Solution.

$$f'(x) = 6(x^2 - 4x + 5)^5 (2x - 4)(2x^3 - 1)^3 + 3(x^2 - 4x + 5)^6 (2x^3 - 1)^2 (6x^2)$$

(c) $g(x) = \frac{3}{(x - \sin(x))^2}$,

Solution.

$$g'(x) = \frac{-6}{(x - \sin(x))^3} \cdot (1 - \cos(x))$$

(d) $f(x) = \frac{\sin(x)\tan(3x)}{1+\cos^2(x)},$

Solution.

$$f'(x) = \frac{(\cos(x)\tan(3x) + 3\sin(x)\sec^2(3x))(1 + \cos^2(x)) + 2\sin^2(x)\tan(3x)\cos(x)}{(1 + \cos^2(x))^2}$$

(e) $h(x) = \sin^3(x^3).$

Solution.

$$h'(x) = 9x^2 \sin^2(x^3) \cos(x^3)$$

3. Evaluate the limit

$$\lim_{h \rightarrow 0} \frac{\tan(\pi/4 + h) - 1}{h}.$$

Solution. We recognize this limit as the derivative of the function $f(x) = \tan(x)$ at $x = \pi/4$. We know that $f'(x) = \sec^2(x)$, so

$$\lim_{h \rightarrow 0} \frac{\tan(\pi/4 + h) - 1}{h} = \sec^2(\pi/4) = \frac{1}{\cos^2(\pi/4)} = \frac{1}{(\sqrt{2}/2)^2} = 2.$$

4. If $f(x) = x^3 - 2x + 1$,

(a) What is the differential of $f(x)$ at $x = 2$?

Solution.

$$df = f'(x)dx = (3x^2 - 2)dx.$$

(b) $f(2) = 5$. Use the differential to approximate $f(1.9)$.

Solution. Note that $dx = 1.9 - 2 = -0.1$ and $df = (3 \cdot 2^2 - 2) \cdot dx = 10 \cdot (-0.1) = -1$. Therefore

$$f(1.9) \approx f(2) + df = 5 - 1 = 4.$$

5. Use implicit differentiation to determine the equation of the tangent line to the curve

$$x^3y^2 - x^2y^3 + 2x^2 - y^2 = 1$$

at the point $(1, 1)$.

Solution. Differentiating both sides,

$$3x^2y^2 + 2x^3yy' - 2xy^3 - 3x^2y^2y' + 4x - 2yy' = 0,$$

and solving yields

$$y' = \frac{-3x^2y^2 + 2xy^3 - 4x}{2x^3y - 3x^2y^2 - 2y}.$$

Evaluating at $(1, 1)$ gives the derivative:

$$y' = \frac{-3 + 2 - 4}{2 - 3 - 2} = \frac{5}{3}$$

Thus the equation of the tangent line is

$$y - 1 = \frac{5}{3}(x - 1).$$

6. A 10m ladder leans against a wall. As the base of the ladder is pulled away at a rate of 0.6m/s, the ladder slides down the wall. At what rate is the height of the top of the ladder changing when the base of the ladder is 6m from the wall? (Give an exact numerical answer)

Solution. If x is the distance that the base has been pulled from the wall, the height of the top of the ladder is $H = \sqrt{100 - x^2}$. Therefore

$$\frac{dH}{dt} = \frac{-x}{\sqrt{100 - x^2}} \frac{dx}{dt}.$$

Evaluating when $x = 6$ and $\frac{dx}{dt} = 0.6$ gives

$$\frac{dH}{dt} = \frac{-6}{\sqrt{100 - 36}} \cdot (0.6) = \frac{-6}{8} \cdot \frac{6}{10} = \frac{-9}{20}.$$

So the height of the top of the ladder is changing at a rate of $-\frac{9}{20} = -0.45\text{m/s}$.

7. Determine where the function $f(x) = x\sqrt{8 - x^2}$ has a horizontal tangent line.

Solution. Differentiate

$$f'(x) = \sqrt{8 - x^2} - x \frac{x}{\sqrt{8 - x^2}},$$

set equal to zero and solve for x :

$$0 = \sqrt{8 - x^2} - \frac{x^2}{\sqrt{8 - x^2}}$$

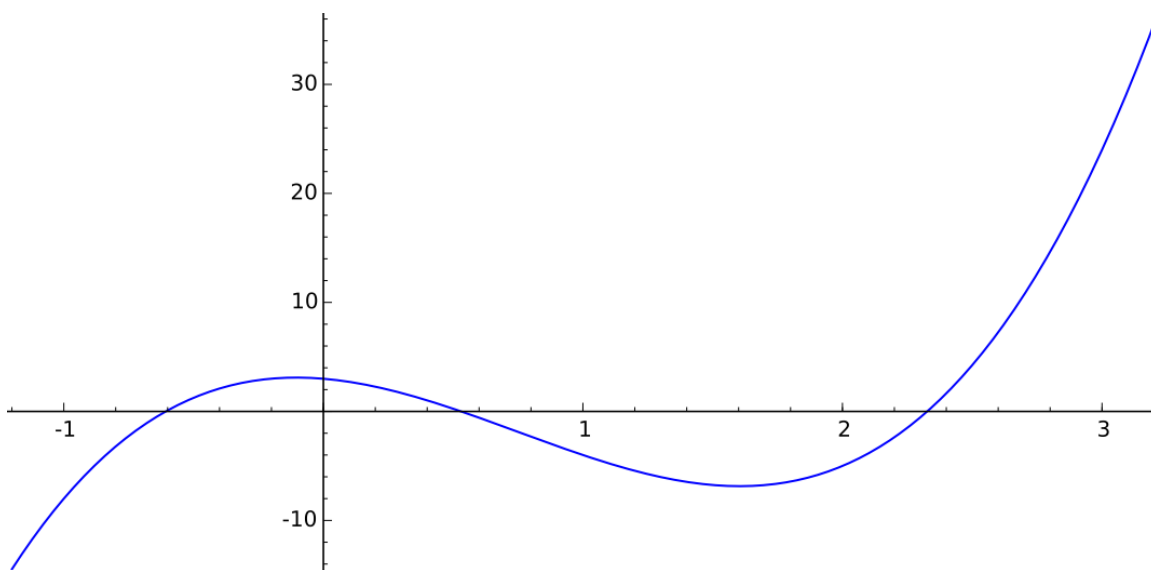
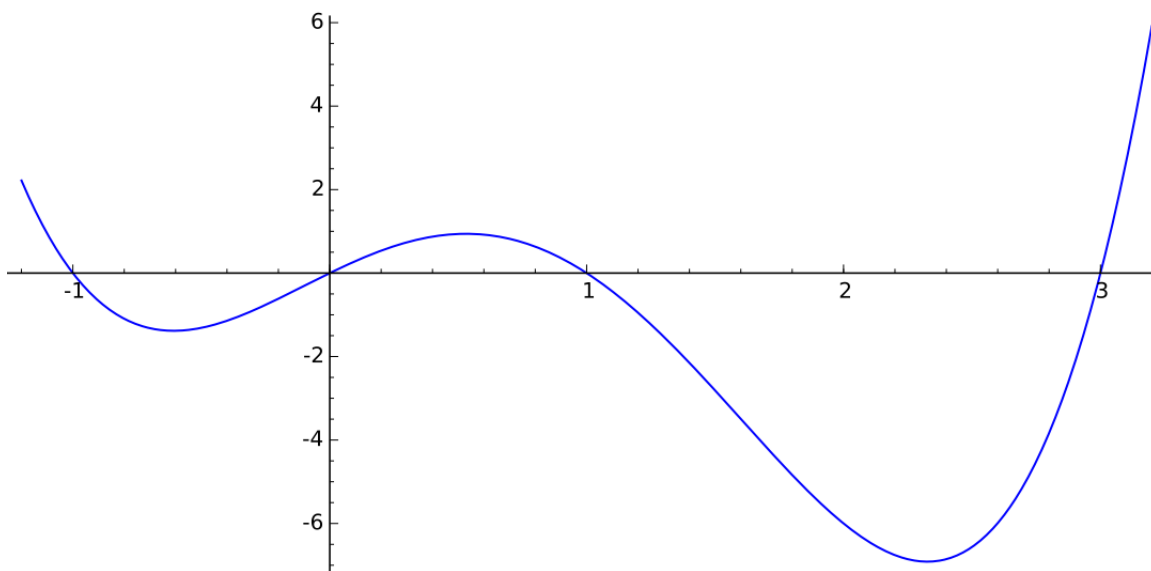
$$0 = (8 - x^2) - x^2$$

$$2x^2 = 8$$

$$x = \pm 2$$

So $f(x)$ has horizontal tangent lines at $x = 2$ and $x = -2$.

8. Sketch the graph of the derivative of the given function $f(x)$:



I would be looking to see that you have the approximate shape correct and that you have the sign of y as a function of x correct (including the locations of the zeros of $f'(x)$). I would not look to check that you had the correct vertical scale (in fact, you might notice that the y scale on this solution graph is different from the blank graph provided).

9. The chart below gives the values of $f(x)$ and $g(x)$ and their derivatives at $x = 2$, $x = 3$ and $x = 4$.

	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	4	-2	3	9
3	0	29	-1	4
4	3	2	7	11

Use it to find the following derivatives,

- (a) If $h(x) = f(g(x))$, determine $h'(4)$.

Solution. By the chain rule,

$$\begin{aligned}h'(4) &= f'(g(4)) \cdot g'(4) \\ &= f'(2) \cdot g'(4) \\ &= 3 \cdot 11 \\ &= 33.\end{aligned}$$

- (b) If $k(x) = g(f(x))$, determine $k'(4)$.

Solution. By the chain rule,

$$\begin{aligned}k'(4) &= g'(f(4)) \cdot f'(4) \\ &= g'(3) \cdot f'(4) \\ &= 4 \cdot 7 \\ &= 28.\end{aligned}$$