Math 220 (7:30pm section) - Exam 1 Solutions

1. Give a value for each of the following limits (including ∞ or $-\infty$ if applicable). (4 points each)

   (a) $\lim_{x\to 2} \frac{x^2-2x}{\sqrt{x+2}-\sqrt{2x}}$

   Solution.

$$\begin{aligned}\lim_{x\to 2} \frac{x^2-2x}{\sqrt{x+2}-\sqrt{2x}} &= \lim_{x\to 2} \frac{x^2-2x}{\sqrt{x+2}-\sqrt{2x}} \cdot \frac{\sqrt{x+2}+\sqrt{2x}}{\sqrt{x+2}+\sqrt{2x}} \\ &= \lim_{x\to 2} \frac{(x-2)x(\sqrt{x+2}+\sqrt{2x})}{x+2-2x} \\ &= \lim_{x\to 2} -x(\sqrt{x+2}+\sqrt{2x}) \\ &= -2\cdot(2+2) = -8.\end{aligned}$$

   (b) $\lim_{\theta\to\pi^-} \cot(\theta)$

   Solution. Since $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$ and since $\lim_{\theta\to\pi^-}\cos(\theta) = -1$ while $\lim_{\theta\to\pi^-}\sin(\theta) = 0$ from above, $\lim_{\theta\to\pi^-}\cot(\theta) = -\infty$.

   (c) $\lim_{t\to 0} \left(\frac{1}{t^2-t} + \frac{1}{t^2+t}\right)$

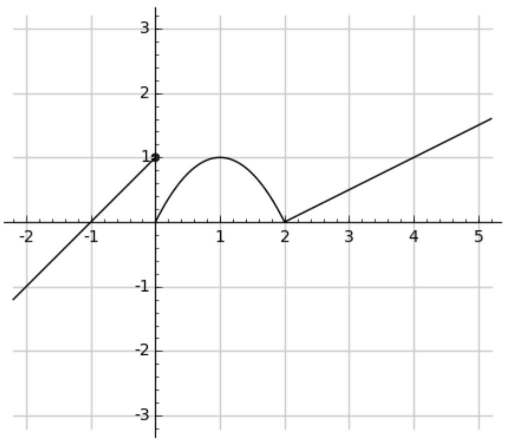
   Solution. Rewriting over a common denominator,

$$\frac{1}{t^2-t} + \frac{1}{t^2+t} = \frac{1}{t}\frac{(t+1)-(t-1)}{(t+1)(t-1)} = \frac{2}{(t+1)(t-1)},$$

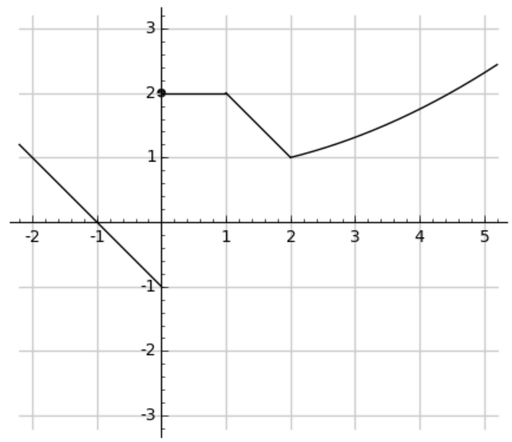
   we see that the limit is $\frac{2}{(1)(-1)} = -2$.

2. Consider the functions $f(x)$ and $g(x)$ shown below:

$f(x)$



$g(x)$



(a) Find $\lim_{x\to 2} g(f(x))$. (4 points)

Solution. We compute the two one-sided limits. As x approaches 2 from above, $f(x)$ approaches 0 from above, so $g(f(x))$ approaches 2. Similarly, as x approaches 2 from below, $f(x)$ still approaches 0 from above, so $g(f(x))$ approaches 2. Thus the limit is 2.

(b) Where is $f(x)$ continuous? (1 point)

Solution. By visual inspection, $f(x)$ is continuous except at $x = 0$.

(c) Where is $g(x)$ continuous? (1 point)

Solution. Similarly, $g(x)$ is continuous except at $x = 0$.

(d) Where is $g(f(x))$ continuous? Justify your answer. (6 points)

Solution. Since the composition of continuous functions is continuous, $g(f(x))$ will be continuous except possibly when $x = 0$ (when $f(x)$ is not continuous) or when $f(x) = 0$ (since $g(x)$ is not continuous at 0). This latter case occurs when $x = 2$, $x = 0$ and $x = -1$. We've already found that $\lim_{x\to 2} g(f(x)) = 2 = g(f(2))$, so $g(f(x))$ is continuous at 2. We similarly compute the following one sided limits:

$$\begin{aligned}
\lim_{x\to 0^+} g(f(x)) &= \lim_{f\to 0^+} g(f) = 2 \\
\lim_{x\to 0^-} g(f(x)) &= \lim_{f\to 1^-} g(f) = 2 \\
\lim_{x\to -1^+} g(f(x)) &= \lim_{f\to 0^+} g(f) = 2 \\
\lim_{x\to -1^-} g(f(x)) &= \lim_{f\to 0^-} g(f) = -1.
\end{aligned}$$

Since $g(f(0)) = 2$, we have that $g(f(x))$ is continuous at $x = 0$ as well, but not at $x = -1$ (since the two one-sided limits are distinct). So, in summary, $g(f(x))$ is continuous everywhere except at $x = -1$.

3. Determine the derivatives of the following functions. (4 points each)

(a) $f(x) = (\sin((1 + x)^7))^3$

Solution. Using the chain rule repeatedly,

$$f'(x) = 3\sin^2((1 + x)^7) \cdot \cos((1 + x)^7) \cdot 7(1 + x)^6.$$

(b) $f(x) = x\sin(x)\sqrt[3]{1+x}$

Solution. Using the product rule repeatedly,

$$f'(x) = \sin(x)\sqrt[3]{1+x} + x\cos(x)\sqrt[3]{1+x} + \frac{x\sin(x)}{3\sqrt[3]{(1+x)^2}}.$$

(c) $f(x) = \dfrac{x+\tan(x)}{1-\cos^2(x)}$

Solution. Using the quotient rule and chain rule,

$$f'(x) = \frac{(1+\sec^2(x))(1-\cos^2(x)) - (x+\tan(x))(2\cos(x)\sin(x))}{(1-\cos^2(x))^2}.$$

(d) $f(x) = x^2\sin\left(\dfrac{1}{x}\right)$

Solution. Using the product and chain rule,

$$f'(x) = 2x\sin\left(\frac{1}{x}\right) + x^2\cos\left(\frac{1}{x}\right)\cdot\frac{-1}{x^2}.$$

(e) $f(x) = (x^3+2)^5(x^2-2)^{-7/2}$

Solution. Using the product and chain rule,

$$f'(x) = 5(x^3+2)^4\cdot(3x^2)\cdot(x^2-2)^{-7/2} - \frac{7}{2}(x^3+2)^5\cdot(x^2-2)^{-9/2}\cdot(2x).$$

4. Find an equation for the tangent line to the curve

$$y\cos(x) = 2+\sin(xy)$$

at the point $(0,2)$. (10 points)

Solution. Using implicit differentiation, we have

$$\begin{aligned} y'\cos(x) - y\sin(x) &= \cos(xy)(y+xy') \\ y'(\cos(x) - x\cos(xy)) &= y(\sin(x)+\cos(xy)) \\ y' &= \frac{y(\sin(x)+\cos(xy))}{\cos(x)-x\cos(xy))}. \end{aligned}$$

Evaluating at $x=0$ and $y=2$ yields $y' = \frac{2(0+1)}{1-0} = 2$. Substituting this slope into the point-slope form of a line yields

$$y = 2x+2.$$

5. A balloon is being filled with air at a rate of $100\frac{\text{cm}^3}{\text{s}}$. Assuming that the balloon is spherical, how fast is its surface area increasing when its radius is 5 cm? (10 points)

Solution. The volume is given by $V = \frac{4}{3}\pi r^3$ and the surface area by $S = 4\pi r^2$. Differentiating both equations with respect to time yields

$$\begin{aligned} V' &= 4\pi r^2 r' \\ S' &= 8\pi r r'. \end{aligned}$$

Substituting $r=5$ and $V'=100$ yields $r' = \frac{1}{\pi}$ from the first equation and then $S' = 40$ from the second. So the surface area is increasing at a rate of $40\frac{\text{cm}^2}{\text{s}}$.

6. Determine the tangent lines to the function $f(x) = \frac{x^2+3x+4}{x-1}$ with slope -1. (10 points)

Solution. We seek values of x with $f'(x) = -1$. Differentiating yields the equation

$$\begin{aligned} f'(x) = \frac{(2x+3)(x-1) - (x^2+3x+4)}{(x-1)^2} &= -1 \\ x^2 - 2x - 7 &= -(x^2 - 2x + 1) \\ 2x^2 - 4x - 6 &= 0 \\ (x-3)(x+1) &= 0. \end{aligned}$$

Evaluating, we have $f(3) = 11$ and $f(-1) = -1$. Thus the two tangent lines with slope -1 are

$$\begin{aligned} y &= -(x-3) + 11, \\ y &= -(x+1) - 1. \end{aligned}$$

7. Suppose that $f(x)$ is a differentiable function, and $h(x) = \sqrt{1 - f(x)}$. If $h(1) = 2$ and $h'(1) = -4$, find $f'(1)$. (10 points)

Solution. Differentiating, we have

$$h'(x) = \frac{-f'(x)}{2\sqrt{1-f(x)}} = \frac{-f'(x)}{2h(x)}$$

by the chain rule. Evaluating at $x = 1$, substituting $h(1) = 2$ and $h'(1) = -4$ and solving, we get

$$\begin{aligned} -4 &= \frac{-f'(1)}{2 \cdot 2} \\ f'(1) &= -16. \end{aligned}$$

8. Let $f(x) = x^9$.

   (a) Find a linear approximation to $f(x)$ near $x = a$. (4 points)

   Solution. The linear approximation equation is

   $$f(x) \approx f(a) + f'(a)(x - a).$$

   Using the definition of $f(x)$, this yields

   $$f(x) \approx a^9 + 9a^8(x - a).$$

   (b) Approximate $(1.01)^9$. (4 points)

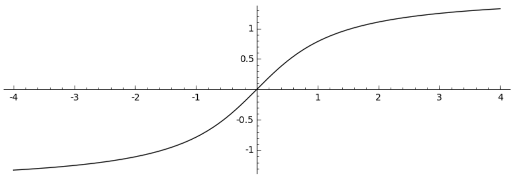
   Solution. We take $x = 1.01$ and $a = 1$ above. This gives

   $$(1.01)^9 = f(1.01) \approx 1^9 + 9 \cdot 1^8 \cdot (0.01) = 1.09.$$

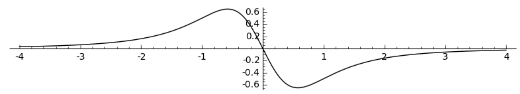
9. Match each graph with its derivative. (2 points per correct match)

$f(x)$ | $f'(x)$

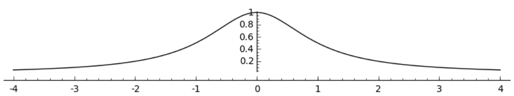
(a)



(1)



(b)



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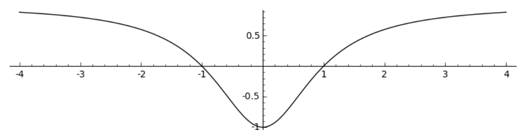
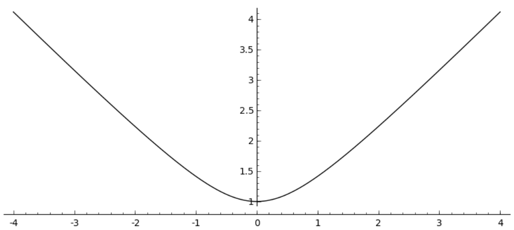


(c)



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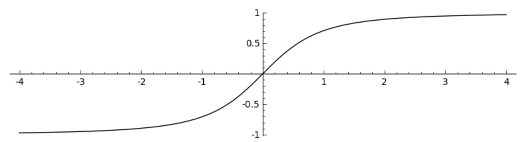

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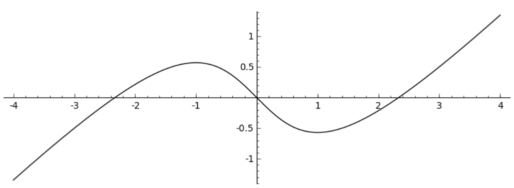

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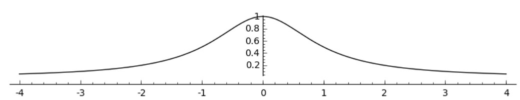

(a) (4)

(b) (1)

(c) (3)

(d) (2)