MATH 0220 EXAM II NAME________________________________

1. Differentiate each of the following. It's not necessary to simplify your answers.

(a) $f(x) = \dfrac{3}{\sqrt{1+x+3x^2}}$

$$f'(x) = -\frac{3(1+6x)}{2(1+x+3x^2)^{3/2}}$$

(b) $s(t) = \arctan(2t)$

$$s'(t) = \frac{2}{1+4t^2}$$

(c) $y = \dfrac{\tan x}{1+\cos x}$

$$y'(x) = \frac{(1+\cos x)\sec^2 x + \tan x \sin x}{(1+\cos x)^2}$$

(d) $g(x) = \ln(x + e^{-3x})$

$$g'(x) = \frac{1-3e^{-3x}}{x+e^{-3x}}$$

(e) $f(x) = \frac{5}{(x^2-4)^3}$

$$f'(x) = -\frac{30x}{(x^2-4)^4}$$

(f) $y = x^{\cos x}$

$$y'(x) = x^{\cos x}\left(-\sin x \ln x + \frac{\cos x}{x}\right)$$

2. (a) Determine the equation of the tangent line to the curve:
$x^2 \sin y + 3x + 2y = 6$ at the point $(2, 0)$.

$$\begin{aligned} 2x\sin x + x^2\cos y\frac{dy}{dx} + 3 + 2\frac{dy}{dx} &= 0 \\ 0 + 4\frac{dy}{dx} + 3 + 2\frac{dy}{dx} &= 0 \\ \frac{dy}{dx} &= -\frac{1}{2} \end{aligned}$$

Therefore, the equation of the tangent line is $y = -\frac{1}{2}(x-2)$

(b) Use your answer in part (a) to approximate y when $x = 2.4$

$$y \approx -0.5(2.4-2) = -0.5(0.4) = -0.2$$

3. Find the point(s) on the parabola $y = x^2$ that is (are) closest to the point $(0, 1)$.

Minimize <u>Distance to (0,1)</u> under constraint <u>point on parabola</u>.

$D = \sqrt{(x-0)^2 + (y-1)^2}$ $\qquad y = x^2$

$$\begin{aligned} D(y) &= \sqrt{y + (y-1)^2} \\ D'(y) &= \frac{1 + 2(y-1)}{2\sqrt{y + (y-1)^2}} \\ 2y - 1 &= 0 \\ y &= \frac{1}{2} \\ x &= \pm\sqrt{\frac{1}{2}} \end{aligned}$$

Points are: $\left(\sqrt{\frac{1}{2}}, \frac{1}{2}\right)$ and $\left(-\sqrt{\frac{1}{2}}, \frac{1}{2}\right)$.

This could have been done replacing y with x^2. Then:

$$\begin{aligned} D(x) &= \sqrt{x^2 + (x^2-1)^2} \\ D'(x) &= \frac{2x + 2(x^2-1)(2x)}{2\sqrt{x^2 + (x^2-1)^2}} \\ 2x + 2(x^2-1)(2x) &= 0 \\ 4x^3 - 2x &= 0 \\ 2x(2x^2 - 1) &= 0 \end{aligned}$$

$x = 0$, $x = \pm\sqrt{\frac{1}{2}}$

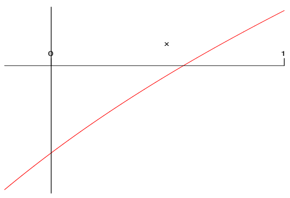
$x = 0$ is a local max. Therefore, the points are $\left(\sqrt{\frac{1}{2}}, \frac{1}{2}\right)$ and $\left(-\sqrt{\frac{1}{2}}, \frac{1}{2}\right)$.

4. A lighthouse is 100 meters from a straight shoreline. The light turns at a rate of 10 revolutions per minute (20π radians/minute), and shines a moving spot of light along the shore. How fast is the spot of light moving when it's 100 meters from the point on the shore which is nearest the lighthouse? Be sure to include units in your answer.

Draw a straight shoreline. Now draw the different points of light on this straight shoreline and label one x. Put the light 100 meters from the shoreline. Draw the light from the lighthouse to some of the points, namely the one labeled x and the one that is exactly 100 meters from the lighthouse. What you have is a right triangle with the adjacent side 100 meters and the opposite side x meters. θ is the angle of rotation of the light. Therefore,

$$\begin{aligned}\tan\theta &= \frac{x}{100}\\ \sec^2\theta\frac{d\theta}{dt} &= \frac{1}{100}\frac{dx}{dt}\\ 2(20\pi)(100)m/min &= \frac{dx}{dt}\end{aligned}$$

5. Use Newton's Method once to estimate the solution to the equation $x = e^{-}x$. (Note that the plot of $f(x) = x - e^{-x}$ is shown below.)



Let $x_0 = 0$. Then $f(x_0) = f(0) = -1$ and since $f'(x) = 1 + e^{-x}$, $f'(x_0) = f'(0) = -1$ and we have:

$$x_1 = 0 - \frac{(-1)}{2} = \frac{1}{2}$$

6. Determine each of the following limits. Show your work.

(a) $\lim_{x \to 0} \frac{\sin x}{e^x - 1} \overset{L'Rule}{=} \lim_{x \to 0} \frac{\cos x}{e^x} = \frac{1}{1} = 1.$

(b) $\lim_{x \to 0} \frac{e^x - 1}{\cos x} = \frac{0}{1} = 0.$

(c) $\lim_{x \to 0} (1 + 2x)^{1/x}$

Let $y = (1 + 2x)^{1/x}$. Then $\ln y = \dfrac{\ln((1 + 2x)}{x}$ and

$\lim_{x \to 0} \ln y = \lim_{x \to 0} \dfrac{\ln(1 + 2x)}{x} \overset{L'Rule}{=} \lim_{x \to 0} \dfrac{\frac{2}{1+2x}}{1} = 2.$

Now if $\ln y \to 2$, then $y \to e^2$