Linear Methods (Math 211) - Lecture 1

David Roe

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Course Information

Instructor	David Roe
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Lecture	**MWF**10-10:50 in **ENE243**
Office Hours	**M**12-14 & **W**11-12 in **MS452**
Labs	**T**16-17 & **W**13-14 & **W**13-14 (register for one)
Tutorials	**M**13-15 & **T**12-16 & **W**13-16 & **R**12-16 in **MS569**
Text	W. Keith Nicholson. *Linear Algebra with Applications.* 6th or 7th ed, McGraw-Hill.

Grading

Assignments	(10%) 10 weekly assignments - `lyryx.com`
Midterm 1	(25%) Tuesday, Oct. 8, 19:00-21:00
Midterm 2	(25%) Monday, Nov. 18, 19:00-21:00
Final	(40%) To be scheduled

Course Outline

1. Solving systems of linear equations
2. Working with matrices, vectors and linear transformations
3. Markov chains
4. Complex numbers
5. Determinants
6. Diagonalization and eigenvalues
7. Linear transformations in geometry

Motivation

- Solving systems of linear equations
- Applications to sciences, social sciences and technology
  - Quantum mechanics, chemistry, electrical engineering, biology, modeling,....
  - Analyzing big data sets
- Applications to other areas of mathematics
  - Number Theory (especially algebraic and computational)
  - Multivariable calculus
  - Other kinds of algebra

Lecture Outline

1. Linear Systems
2. Matrices
3. Geometric Perspective
4. Parametric Form

Linear systems

Linear equation – An equation where variables are only multiplied by numbers: $5x - 17y = 4$, for example. Generally,

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b.$$

Linear system – A finite set of linear equations:

$$\begin{aligned} 5x - 17y &= 4 \\ -3x + 11y &= 8 \end{aligned}$$

for example. Generally,

$$\begin{array}{ccccccccc} a_{11}x_1 &+& a_{12}x_2 &+& \cdots &+& a_{1n}x_n &=& b_1 \\ \vdots & & \vdots & & & & \vdots & & \vdots \\ a_{m1}x_1 &+& a_{m2}x_2 &+& \cdots &+& a_{mn}x_n &=& b_m. \end{array}$$

Solutions to linear systems

A solution to a linear equation is a sequence $(s_1, s_2, \ldots, s_n)$ so that

$$a_1 s_1 + a_2 s_2 + \cdots + a_n s_n;$$

a solution to a linear system is a simultaneous solution for all of its equations.
For example, $(45, 13)$ is a solution to

$$\begin{aligned} 5x - 17y &= 4 \\ -3x + 11y &= 8 \end{aligned}$$

$$\text{since } 5 \cdot 45 - 17 \cdot 13 = 225 - 221 = 4$$

$$\text{and } -3 \cdot 45 + 11 \cdot 13 = -135 + 143 = 8$$

A linear system is consistent if it has a solution and inconsistent otherwise. Consistent systems may have either one solution or infinitely many.

Solving Linear Systems

Linear systems can be solved using two operations:

1. Multiplying an equation by a nonzero number
2. Adding a multiple of one equation to another.

$$\begin{aligned} 5x - 17y &= 4 \\ -3x + 11y &= 8 \end{aligned} \qquad \begin{aligned} x - 3y &= 6 \\ 2y &= 26 \end{aligned}$$

$$\begin{aligned} 2x - 6y &= 12 \\ -3x + 11y &= 8 \end{aligned} \qquad \begin{aligned} x - 3y &= 6 \\ y &= 13 \end{aligned}$$

$$\begin{aligned} x - 3y &= 6 \\ -3x + 11y &= 8 \end{aligned} \qquad \begin{aligned} x \qquad &= 45 \\ y &= 13 \end{aligned}$$

Matrix Perspective

Writing lots of variables gets annoying; we can express any linear system in the form $A \cdot \mathbf{x} = \mathbf{b}$ for a matrix A and a vector $\mathbf{b}$, where $\mathbf{x}$ is a vector of variables. For example,

$$\begin{aligned} 5x - 17y &= 4 \\ -3x + 11y &= 8 \end{aligned}$$

becomes

$$\begin{bmatrix} 5 & -17 \\ -3 & 11 \end{bmatrix} \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$

We will explore matrices and vectors further next week; for now, just remember the process of extracting the coefficient matrix and constant matrix from a linear system.

Augmented Matrix

We can put all of the numbers defining a linear system in a single matrix, called the augmented matrix; we divide the coefficient matrix from the constant matrix with a vertical line:

$$\left[\begin{array}{cc|c} 5 & -17 & 4 \\ -3 & 11 & 8 \end{array}\right]$$

The rows of the augmented matrix correspond to equations in the linear system.

Elementary Row Operations

Since the rows of the augmented matrix correspond to the equations in the linear system, we can perform row operations to solve the system. We may:

1. Multiply a row by a nonzero number,
2. Add a multiple of one row to another row,
3. Swap two rows.

For example, we can solve our example system:

$$\left[\begin{array}{rr|r} 5 & -17 & 4 \\ -3 & 11 & 8 \end{array}\right] \to \left[\begin{array}{rr|r} 2 & -6 & 12 \\ -3 & 11 & 8 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & 6 \\ -3 & 11 & 8 \end{array}\right]$$

$$\to \left[\begin{array}{rr|r} 1 & -3 & 6 \\ 0 & 2 & 26 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & -3 & 6 \\ 0 & 1 & 13 \end{array}\right] \to \left[\begin{array}{rr|r} 1 & 0 & 45 \\ 0 & 1 & 13 \end{array}\right]$$

Inverting Row Operations

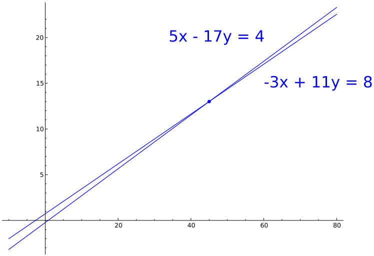
Each of the three types of row operations can be undone:

1. If you multiply a row by a nonzero number α, you can invert this operation by multiplying the same row by α^{-1}.
2. If you add α times row i to row j, adding $-\alpha$ times row i to row j will return the matrix to its original state.
3. If you swap two rows then swapping them again yields the matrix you started with.

We say that two linear systems are equivalent if they have the same set of solutions. Since any row operation can be inverted, two linear systems that are related by a sequence of row operations are equivalent (Theorem 1).

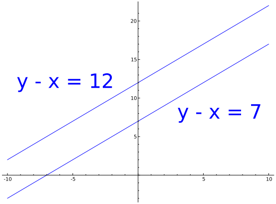
Geometric Perspective

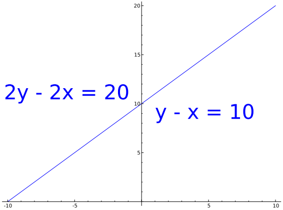
Linear equations in two variables correspond yield lines in the plane, and we can interpret the solutions geometrically.



How many solutions

If the lines are parallel, then there will be either no solutions or infinitely many:





We would like to be able to transform a linear system so that its solutions are immediately apparent. For example, suppose

$$3x + 4y + 5z = 29.$$

We could divide by 5 and rearrange to get

$$z = 29 - \frac{3}{5}x - \frac{4}{5}y$$

If we set

$$\begin{aligned} x &= s \\ y &= t \\ z &= 29 - \frac{3}{5}s - \frac{4}{5}t \end{aligned}$$

then *any* values of s and t will produce a solution.

Uniqueness of Parametric Form

Note that the description of the solutions in parametric form is not unique: we could have solved for y instead of z:

$$\begin{aligned} x &= s \\ y &= 29 - \frac{3}{4}s - \frac{5}{4}t \\ z &= t \end{aligned}$$

Lecture Summary

1. Linear Systems
2. Matrices
3. Geometric Perspective
4. Parametric Form

Next Time

We'll learn a general process for finding a parametric form for a linear system, called Gaussian elimination.