Linear Methods (Math 211)
Lecture 10 - §2.4

(with slides adapted from K. Seyffarth)

David Roe

September 30, 2013

Recall

1. More Block Multiplication
2. Matrix Inverses for 2×2 Matrices

Today

Exam Information

- The first midterm will be held **7pm to 9pm** on **Tuesday, October 8** in **ICT 102**.
- If you have a conflict with this time you should notify me immediately by e-mail (roed.math@gmail.com).
- If you are sick and can't attend the exam, e-mail me **before the exam**.

Exam Regulations

- Calculators and other electronic devices are **not** permitted.
- You should bring your **student ID** or other form of identification.
- There will be a TA in the room available to answer questions.
- Obviously, collaboration and copying are not permitted.
- See the University Calendar (section G) for more details.

Exam Topics

- §1.1 Solutions and Elementary Operations
- §1.2 Gaussian Elimination
- §1.3 Homogeneous Equations
- §2.1 Matrix Addition, Scalar Multiplication, and Transposition
- §2.2 Equations, Matrices, and Transformations
- §2.3 Matrix Multiplication (no Block Multiplication or Directed Graphs)
- §2.4 Matrix Inverses (up to but NOT including Inverses of Matrix Transformations)

Review Opportunities

- I will hold an hour-long, **in-class** review either **Friday or Monday** (class vote)
- Calgary SOS (student group) is running a review session **5pm-8pm** on **Friday, October 4** in **ST 128**. It costs **$20**.
- There are two practice midterms posted on Blackboard. Solutions will be posted on Friday.
- Your lab this week is a good place to ask questions.
- You can also get help in continuous tutorial in **MS 569**
- I will have office hours **Monday 12-2** in **MS 452**. I am permanently moving my Wednesday office hours to **Friday**, still 11-12.

The Matrix Inversion Algorithm

Example

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists.

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Let $A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & -1 & 3 \\ 1 & 2 & 4 \end{bmatrix}$. Find the inverse of A, if it exists.

Begin with a matrix obtained from A by **augmenting** A with the 3×3 identity matrix. We write this as

$$\left[A \mid I \right] = \left[\begin{array}{rrr|rrr} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right]$$

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Then perform elementary row operations on this matrix until the **left** half of the matrix is transformed into I_3.

The Matrix Inversion Algorithm

Example (continued)

$$\left[\begin{array}{ccc|ccc} 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & -1 & 3 & 0 & 1 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 3 & 1 & 2 & 1 & 0 & 0 \\ 1 & 2 & 4 & 0 & 0 & 1 \end{array}\right] \to$$

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$$\left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 4 & -7 & 1 & -3 & 0 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & -1 & 3 & 0 & 1 & 0 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 3 & 1 & 0 & -1 & 1 \end{array}\right] \rightarrow$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 25 & -3 & 5 & 4 \end{array}\right] \rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -5 & 1 & -1 & -1 \\ 0 & 1 & -8 & 1 & -2 & -1 \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right] \rightarrow$$

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$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{10}{25} & 0 & -\frac{5}{25} \\ 0 & 1 & 0 & \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ 0 & 0 & 1 & -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{array}\right]$$

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$$\left[A \mid I\right] \to \left[I \mid A^{-1}\right].$$

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We have successfully transformed the left-hand side of $\left[A \mid I\right]$ into I using elementary row operations, and thus

$$\left[A \mid I\right] \to \left[I \mid A^{-1}\right].$$

Therefore, A^{-1} exists, and

$$A^{-1} = \begin{bmatrix} \frac{10}{25} & 0 & -\frac{5}{25} \\ \frac{1}{25} & -\frac{10}{25} & \frac{7}{25} \\ -\frac{3}{25} & \frac{5}{25} & \frac{4}{25} \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 10 & 0 & -5 \\ 1 & -10 & 7 \\ -3 & 5 & 4 \end{bmatrix}$$

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You can check your work by computing AA^{-1} and $A^{-1}A$.

Justification

- If A is invertible then $A\mathbf{x} = \mathbf{b}$ has a **unique** solution $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b}$.
- Suppose $\mathbf{b} = \mathbf{e}_j = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \end{bmatrix}^T$. Then $A^{-1}\mathbf{b}$ is the jth column of A^{-1}.
- We can also solve the equation $A\mathbf{x} = \mathbf{e}_j$ with Gaussian elimination: row reduce the augmented matrix $\left[A \mid \mathbf{e}_j\right]$.
- All of the row reduction steps depend only on A, so we can save time by doing them all at once, and using the augmented matrix $\left[A \mid I\right]$.

Theorem (§2.4 Theorem 3)

Let A be an $n \times n$ matrix. If A can be transformed to I_n using elementary row operations, then A is invertible and the matrix inversion algorithm produces A^{-1}. Otherwise, A is not invertible.

Example

Find, if possible, the inverse of $\begin{bmatrix} 1 & 0 & -1 \\ -2 & 1 & 3 \\ -1 & 1 & 2 \end{bmatrix}$.

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$$\rightarrow \left[\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 1 \end{array}\right]$$

Since the reduced row echelon form doesn't have an identity matrix on the left, we see that A has no inverse.

Cancellation Laws

Example (§2.4 Example 7)

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$$\begin{aligned} A^{-1}(AB) &= A^{-1}(AC) \\ (A^{-1}A)B &= (A^{-1}A)C \\ IB &= IC \\ B &= C. \end{aligned}$$

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2. If $BA = CA$, then

$$\begin{aligned} (BA)A^{-1} &= (CA)A^{-1} \\ B(AA^{-1}) &= C(AA^{-1}) \\ BI &= CI \\ B &= C. \end{aligned}$$

Problem

Find matrices A, B *and* C *for which* $AB = AC$ *but* $B \neq C$.

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3. *If* A *and* B *are invertible, so is* AB*, and* $(AB)^{-1} = B^{-1}A^{-1}$.
4. *If* $A_1, A_2, \ldots, A_k$ *are invertible, so is* $A_1 A_2 \cdots A_k$*, and* $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$.

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4. *If $A_1, A_2, \ldots, A_k$ are invertible, so is $A_1 A_2 \cdots A_k$, and $(A_1 A_2 \cdots A_k)^{-1} = A_k^{-1} A_{k-1}^{-1} \cdots A_2^{-1} A_1^{-1}$.*
5. *If A is invertible, so is A^k, and $(A^k)^{-1} = (A^{-1})^k$.*

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5. *If A is invertible, so is A^k, and $(A^k)^{-1} = (A^{-1})^k$.*
6. *If A is invertible and $a \in \mathbb{R}$ is nonzero, then aA is invertible, and $(aA)^{-1} = \frac{1}{a}A^{-1}$.*

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Assume all matrices are $n \times n$.

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5. *If A is invertible, so is A^k, and $(A^k)^{-1} = (A^{-1})^k$.*
6. *If A is invertible and $a \in \mathbb{R}$ is nonzero, then aA is invertible, and $(aA)^{-1} = \frac{1}{a}A^{-1}$.*
7. *If A is invertible, so is A^T, and $(A^T)^{-1} = (A^{-1})^T$.*

Example

True or false? If $A^3 = 4I$, then A is invertible.

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If $A^3 = 4I$, then

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so

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Therefore A is invertible, and $A^{-1} = \frac{1}{4}A^2$. **True**

Example

True or false? If A and B are invertible, then $A + B$ is invertible.

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Take $A = I$ and $B = -I$. Both are invertible but $A + B = 0$ is not.
False

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Corollary

If A and C are $n \times n$ matrices such that $AC = I$, then $CA = I$ and $C = A^{-1}$, $A = C^{-1}$.

In the Corollary, it is essential that the matrices be square.

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Example

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad CA = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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Suppose B is the inverse of A^2. Then

$$A^2 B = A(AB) = I$$

Therefore AB is the inverse of A. **True**

Summary

1. Exam Announcements
2. The Matrix Inversion Algorithm
3. Properties of Inversion