Linear Methods (Math 211)
Lecture 9 - §2.3 & 2.4

(with slides adapted from K. Seyffarth)

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Recall

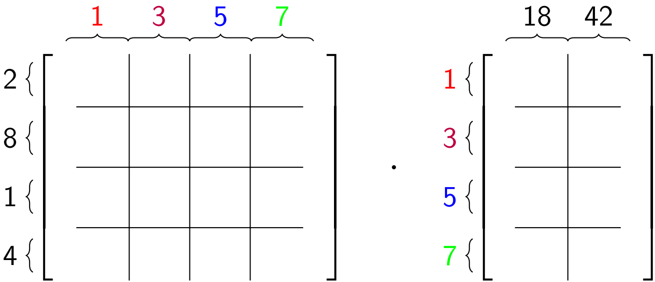
1. Properties of Matrix Multiplication
2. Block Multiplication

Today

Compatibility of Blocks

A division of A and B into blocks is compatible if

- The number of *block columns* of A is equal to the number of *block rows* of B,
- The width of each block column of A is the same as the height of the corresponding block row of B.



Strassen multiplication

Suppose A and B are large $2n \times 2n$ matrices. Divide each into $n \times n$ blocks:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \cdot \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right]$$

Suppose multiplying $2n \times 2n$ matrices takes $2(2n)^3 = 16n^3$ operations. We've replaced this with eight multiplications, each taking $2n^3$ operations. No benefit!

Strassen multiplication

Define

$$\begin{aligned} M_1 &= (A_{11}+A_{22})(B_{11}+B_{22}) & M_2 &= (A_{21}+A_{22})B_{11} \\ M_3 &= A_{11}(B_{12}-B_{22}) & M_4 &= A_{22}(B_{21}-B_{11}) \\ M_5 &= (A_{11}+A_{12})B_{22} & M_6 &= (A_{21}-A_{11})(B_{11}+B_{12}) \\ M_7 &= (A_{12}-A_{22})(B_{21}+B_{22}) \end{aligned}$$

Then

$$A \cdot B = \left[\begin{array}{c|c} M_1+M_4-M_5+M_7 & M_3+M_5 \\ \hline M_2+M_4 & M_1-M_2+M_3+M_6 \end{array}\right]$$

We've replaced 8 multiplications and 4 additions with 7 multiplications and 18 additions. Even better, we can recurse, and use the same technique to multiply the $n \times n$ blocks. The resulting algorithm takes about $n^{2.807}$ operations.

Matix Inverses

Definition

Let A be an $n \times n$ matrix. Then B is an inverse of A if and only if $AB = I_n$ and $BA = I_n$.

Since A and I_n are both $n \times n$, B **must also be** an $n \times n$ matrix.

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Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$. Then

$$AB = ?$$

and

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Example

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$. Then

$$AB = I_2$$

and

$$BA = I_2$$

so B is an inverse of A.

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Example

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

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has no inverse.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a+b \\ 0 & c+d \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem (§2.4 Theorem 1)

If B and C are inverses of A, then $B = C$.

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Proof.

We have

$$B = BI = B(AC) = (BA)C = IC = C.$$

□

Let A be a square matrix, i.e., an $n \times n$ matrix.

- The inverse of A, if it exists, is denoted A^{-1}, and

$$AA^{-1} = I = A^{-1}A.$$

- If A has an inverse, then we say that A is invertible.

The inverse of a 2×2 matrix

Example (§2.4 Example 4)

Suppose that $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then A is invertible if and only if

$$ad - bc \neq 0.$$

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$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

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$$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cdot \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix}$$

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

- $ad - bc$ is the determinant of A, and is denoted $\det A$.
- $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ is the adjugate of A, and is denoted $\operatorname{adj} A$.

We will eventually generalize both to $n \times n$ matrices.

Systems of Linear Equations and Inverses

Suppose that a system of n linear equations in n variables is written in matrix form as $A\mathbf{x} = \mathbf{b}$, and suppose that A is invertible.

Example

The system of linear equations

$$\begin{aligned} 2x - 7y &= 3 \\ 5x - 18y &= 8 \end{aligned}$$

can be written in matrix form as $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

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Here A is invertible since $2(-18) - 5(-7) = -1 \neq 0$.

Since A is invertible, A^{-1} exists and has the property that $AA^{-1} = I = A^{-1}A$, and thus

$$\begin{aligned} A\mathbf{x} &= \mathbf{b} \\ A^{-1}(A\mathbf{x}) &= A^{-1}\mathbf{b} \\ (A^{-1}A)\mathbf{x} &= A^{-1}\mathbf{b} \\ I\mathbf{x} &= A^{-1}\mathbf{b} \\ \mathbf{x} &= A^{-1}\mathbf{b}, \end{aligned}$$

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i.e., $A\mathbf{x} = \mathbf{b}$ has the **unique solution** given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Example (continued)

Recall that we have the system $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

- $\det A =$
- $\text{adj } A = \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$
- $A^{-1} =$.

Example (continued)

Recall that we have the system $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & -7 \\ 5 & -18 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}.$$

- $\det A = -1$
- $\text{adj } A = \begin{bmatrix} -18 & 7 \\ -5 & 2 \end{bmatrix}$
- $A^{-1} = \frac{1}{\det A} \text{adj } A = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix}.$
- Therefore,

$$\mathbf{x} = A^{-1}\mathbf{b} = \begin{bmatrix} 18 & -7 \\ 5 & -2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 8 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$$

If A is a 2×2 matrix, then it is easy to determine if A is invertible: compute $\det A$.

If $\det A \neq 0$, find $\text{adj}\ A$; then

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Problem

Suppose that A is a 3×3 matrix, or, more generally, an $n \times n$ matrix where $n \geq 3$.

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- *If A^{-1} exists, how do we find it?*

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Answer: the matrix inversion algorithm.

Summary

1. More Block Multiplication
2. Matrix Inverses