Linear Methods (Math 211)
Lecture 8 - §2.3

(with slides adapted from K. Seyffarth)

David Roe

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Recall

1. Matrix Transformations
2. Matrix Multiplication
3. Commutativity of Matrix Multiplication

Today

1. Properties of Matrix Multiplication
2. Block Multiplication

Theorem (§2.3 Theorem 3)

Let A, B, and C be matrices of appropriate sizes, and let $k \in \mathbb{R}$ be a scalar.

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1. $IA = A$ and $AI = A$ where I is an identity matrix.
2. $A(BC) = (AB)C$ (associative property)
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5. $k(AB) = (kA)B = A(kB)$

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4. $(B + C)A = BA + CA$ (distributive property)
5. $k(AB) = (kA)B = A(kB)$
6. $(AB)^T = B^T A^T$

Example (§2.3 Example 7)

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$$\begin{aligned}
&A(BC - CD) + A(C - B)D - AB(C - D) \\
&\qquad = A(BC) - A(CD) + (AC - AB)D - (AB)C + (AB)D \\
&\qquad = ABC - ACD + ACD - ABD - ABC + ABD \\
&\qquad = 0
\end{aligned}$$

Scalar Matrices

A matrix of the form aI_n is called a scalar matrix:

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}.$$

Scalar matrices commute with any $n \times n$ matrix B:

$$(aI_n)B = aB = B(aI_n).$$

For example,

$$\begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$$

Diagonal Matrices

More generally, a diagonal matrix is a matrix where the only nonzero entries are on the diagonal:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$$

The product of two diagonal matrices is another diagonal matrix. Diagonal matrices commute with **each other**, but generally not other matrices.

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 14 & 0 \\ 0 & 15 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 15 & 20 \end{bmatrix} \neq \begin{bmatrix} 2 & 10 \\ 6 & 20 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}$$

Elementary Proofs

Example

Let A and B be $m \times n$ matrices, and let C be an $n \times k$ matrix. Prove that if A and B commute with C, then $A + B$ commutes with C.

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Proof.

We are given that $AC = CA$ and $BC = CB$. Consider $(A + B)C$.

$$\begin{aligned}(A+B)C &= AC + BC \\ &= CA + CB \\ &= C(A+B)\end{aligned}$$

Since $(A + B)C = C(A + B)$, $A + B$ commutes with C. □

Elementary Proofs 2

Example

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Proof.

Suppose AB is symmetric. Then

$$AB = (AB)^T = B^T A^T = BA.$$

Conversely, if $AB = BA$ then

$$(AB)^T = B^T A^T = BA = AB,$$

so AB is symmetric. □

Block Multiplication

Example

Let A be an $m \times n$ matrix. Let B be an $n \times k$ matrix with columns $B_1, B_2, \ldots, B_k$, i.e., $B = \begin{bmatrix} B_1 & B_2 & \cdots & B_k \end{bmatrix}$. This represents a partition of B into blocks – in this example, the blocks are the columns of B. We can now write

$$\begin{aligned} AB &= A \begin{bmatrix} B_1 & B_2 & \cdots & B_k \end{bmatrix} \\ &= \begin{bmatrix} AB_1 & AB_2 & \cdots & AB_k \end{bmatrix} \end{aligned}$$

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Here, the columns of AB, namely $AB_1, AB_2, \ldots, AB_k$, can be thought of as blocks of AB.

If A is an $m \times n$ matrix and B is an $n \times k$ matrix, and if A and B are partitioned compatibly into blocks in some way, then the computation of the product AB may be simplified.

Example

$$A = \begin{bmatrix} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

Example

$$A = \left[\begin{array}{cc|cc} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \qquad B = \begin{bmatrix} 1 & 2 & 0 \\ -1 & 0 & 0 \\ 0 & 5 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$

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Example (continued)

Let

$$A = \left[\begin{array}{cc|cc} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ 0 & I_2 \end{bmatrix},$$

and let

$$B = \left[\begin{array}{cc|c} 1 & 2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right] = \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix}.$$

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Then

$$AB = \begin{bmatrix} A_1 & A_2 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix}.$$

Example (continued)

$$
\begin{aligned}
AB &= \begin{bmatrix} A_1 & A_2 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix} \\
&= \begin{bmatrix} A_1 \cdot B_1 + A_2 \cdot B_2 & A_1 \cdot 0 + A_2 \cdot B_3 \\ 0 \cdot B_1 + I_2 \cdot B_2 & 0 \cdot 0 + I_3 \cdot B_3 \end{bmatrix} \\
&= \begin{bmatrix} A_1 B_1 + A_2 B_2 & A_2 B_3 \\ B_2 & B_3 \end{bmatrix}
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Example (continued)

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\end{aligned}
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Recall that

$$
A = \left[\begin{array}{rr|rr} 2 & -1 & 3 & 1 \\ 1 & 0 & 1 & 2 \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = \begin{bmatrix} A_1 & A_2 \\ 0 & I_2 \end{bmatrix}, B = \left[\begin{array}{rr|r} 1 & 2 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right] = \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix}.
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Example (continued)

$$\begin{aligned} AB &= \begin{bmatrix} A_1 & A_2 \\ 0 & I_2 \end{bmatrix} \begin{bmatrix} B_1 & 0 \\ B_2 & B_3 \end{bmatrix} \\ &= \begin{bmatrix} A_1 \cdot B_1 + A_2 \cdot B_2 & A_1 \cdot 0 + A_2 \cdot B_3 \\ 0 \cdot B_1 + I_2 \cdot B_2 & 0 \cdot 0 + I_3 \cdot B_3 \end{bmatrix} \\ &= \begin{bmatrix} A_1 B_1 + A_2 B_2 & A_2 B_3 \\ B_2 & B_3 \end{bmatrix} \end{aligned}$$

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Now compute $A_1 B_1$, $A_2 B_2$ and $A_2 B_3$.

Example (continued)

$$A_1B_1 = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

Example (continued)

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$$A_2 B_2 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 14 \\ 2 & 3 \end{bmatrix}$$

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$$A_2B_3 = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

Now,

$$AB = \begin{bmatrix} A_1B_1 + A_2B_2 & A_2B_3 \\ B_2 & B_3 \end{bmatrix} = \left[\begin{array}{cc|c} 4 & 18 & 3 \\ 3 & 5 & 1 \\ \hline 0 & 5 & 1 \\ 1 & -1 & 0 \end{array}\right] = \begin{bmatrix} 4 & 18 & 3 \\ 3 & 5 & 1 \\ 0 & 5 & 1 \\ 1 & -1 & 0 \end{bmatrix}$$