Linear Methods (Math 211)
Lecture 7 - §2.2 & §2.3

(with slides adapted from K. Seyffarth)

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Recall

1. Associated Homogeneous Systems
2. Matrix Transformations

Today

1. More Matrix Transformations
2. Matrix Multiplication
3. Commutativity of Matrix Multiplication

Zero and One

1. If A is the $m \times n$ matrix of all zeros, then the transformation induced by A, namely

$$T(\mathbf{x}) = \mathbf{0} \text{ for all } \mathbf{x} \in \mathbb{R}^n,$$

is called the zero transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and is written $T=0$.

2. If A is the $n \times n$ identity matrix, then the transformation induced by A is called the identity transformation on $\mathbb{R}^n$, and is written $1_{\mathbb{R}^n}$. We have

$$1_{\mathbb{R}^n}(\mathbf{x}) = \mathbf{x} \text{ for all } \mathbf{x} \in \mathbb{R}^n.$$

Polar Coordinates

Recall that we can specify a vector $\begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2$ either in rectangular coordinates or in polar coordinates. In polar coordinates, we specify the distance r from the origin and the angle θ from the positive x-axis. The two systems of coordinates are related by

$$a = r\cos(\theta)$$
$$b = r\sin(\theta)$$

$$r = \sqrt{a^2 + b^2}$$
$$\theta = \begin{cases} \tan^{-1}(\frac{b}{a}) & \text{if } a > 0 \\ \pi + \tan^{-1}(\frac{b}{a}) & \text{if } a < 0 \end{cases}$$

Rotations

Suppose we want to rotate counterclockwise by an angle α. Then a point with polar coordinates (r, θ) will map to one with polar coordinates $(r, \theta + \alpha)$. In rectangular coordinates, this means

$$\begin{aligned} \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix} &\mapsto \begin{bmatrix} r\cos(\theta+\alpha) \\ r\sin(\theta+\alpha) \end{bmatrix} \\ &= \begin{bmatrix} r\cos(\theta)\cos(\alpha) - r\sin(\theta)\sin(\alpha) \\ r\sin(\theta)\cos(\alpha) + r\cos(\theta)\sin(\alpha) \end{bmatrix} \\ &= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} r\cos(\theta) \\ r\sin(\theta) \end{bmatrix}, \end{aligned}$$

so rotation by α is a matrix transformation, induced by the matrix

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$

Translations

A matrix transformation must map the zero vector $\mathbf{0}$ to itself.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by

$$T(\mathbf{x}) = \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ for all } \mathbf{x} \in \mathbb{R}^2.$$

This translation is not a matrix transformation.

Matrix Multiplication

Let A be an $m \times n$ matrix and $B = \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix}$ an $n \times k$ matrix, whose columns are $\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_k$. The product of A and B is the matrix

$$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_k \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_k \end{bmatrix},$$

i.e., the first column of AB is $A\mathbf{b}_1$, the second column of AB is $A\mathbf{b}_2$, etc.

Note that the number of **columns** of the first matrix must match the number of **rows** of the second.

Example

Let A and B be matrices,

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \text{ and } B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

Then AB has columns

$$A\mathbf{b}_1 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix},$$

$$A\mathbf{b}_2 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$

$$A\mathbf{b}_3 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$

Example (continued)

Putting the columns together, we get

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}.$$

Theorem (§2.3 Theorem 1)

Let A be an $m \times n$ matrix, and B an $n \times k$ matrix. Then

$$A(B\mathbf{x}) = (AB)\mathbf{x} \text{ for all } k\text{-vectors } \mathbf{x} \in \mathbb{R}^k.$$

Proof.

We can write $B\mathbf{x}$ as $x_1\mathbf{b}_1 + \cdots + x_k\mathbf{b}_k$, so

$$A(B\mathbf{x}) = x_1A\mathbf{b}_1 + \cdots + x_kA\mathbf{b}_k.$$

On the other hand, the columns of AB are $A\mathbf{b}_1, \ldots, A\mathbf{b}_k$, so we have

$$(AB)\mathbf{x} = x_1A\mathbf{b}_1 + \cdots + x_kA\mathbf{b}_k.$$

□

Transformations and Matrix Multiplication

Suppose that $\mathbb{R}^k \xrightarrow{T_B} \mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n$ are matrix transformations, induced by an $n \times m$ matrix A and an $m \times k$ matrix B. The composite of T_A and T_B, written $T_A \circ T_B$, is defined by $(T_A \circ T_B)(\mathbf{x}) = T_A(T_B(\mathbf{x}))$ for $\mathbf{x} \in \mathbb{R}^k$.

The theorem implies that the composite of two matrix transformations is a matrix transformation, induced by AB.

Any *invertible* matrix transformation of $\mathbb{R}^2$ can be written as a composite of shears, reflections and x or y-expansions.

Dot Products and Matrix Multiplication

Theorem (§2.3 Theorem 2)

Let A be an $m \times n$ matrix and B and $n \times k$ matrix. Then the (i, j)-entry of AB is the dot product of row i of A with column j of B.

Dot Products and Matrix Multiplication

Theorem (§2.3 Theorem 2)

Let A be an $m \times n$ matrix and B and $n \times k$ matrix. Then the (i, j)-entry of AB is the dot product of row i of A with column j of B.

Example

Use the above theorem to compute

$$\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \cdot \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

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Compatible Sizes

- Let A and B be matrices. In order for the product AB to exist, the number of rows in B must be equal to the number of columns in A.
- Assuming that A is an $m \times n$ matrix, the product AB is defined if and only if B is an $n \times k$ matrix for some k. If the product is defined, then A and B are said to be compatible for (matrix) multiplication.
- Given that A is $m \times n$ and B is $n \times k$, the product AB is an $m \times k$ matrix.

Example (revisited)

As we saw earlier,

$$\overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}} \overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} = \overset{2\times 3}{\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}}$$

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As we saw earlier,

$$\overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}} \overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} = \overset{2\times 3}{\begin{bmatrix} 4 & -1 & -2 \\ -1 & 4 & 0 \end{bmatrix}}$$

Note that the product

$$\overset{3\times 3}{\begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}} \overset{2\times 3}{\begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix}}$$

does not exist.

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

- Does AB exist? If so, compute it.
- Does BA exist? If so, compute it.

Example

Let

$$A=\begin{bmatrix}1 & 2\\ -3 & 0\\ 1 & -4\end{bmatrix} \text{ and } B=\begin{bmatrix}1 & -1 & 2 & 0\\ 3 & -2 & 1 & -3\end{bmatrix}$$

- Does AB exist? If so, compute it.
- Does BA exist? If so, compute it.

$$AB=\begin{bmatrix}7 & -5 & 4 & -6\\ -3 & 3 & -6 & 0\\ -11 & 7 & -2 & 12\end{bmatrix}$$

Example

Let

$$A = \begin{bmatrix} 1 & 2 \\ -3 & 0 \\ 1 & -4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 1 & -1 & 2 & 0 \\ 3 & -2 & 1 & -3 \end{bmatrix}$$

- Does AB exist? If so, compute it.
- Does BA exist? If so, compute it.

$$AB = \begin{bmatrix} 7 & -5 & 4 & -6 \\ -3 & 3 & -6 & 0 \\ -11 & 7 & -2 & 12 \end{bmatrix}$$

BA does not exist

Example

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does GH exist? If so, compute it.
- Does HG exist? If so, compute it.

Example

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does GH exist? If so, compute it.
- Does HG exist? If so, compute it.

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

Example

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- Does GH exist? If so, compute it.
- Does HG exist? If so, compute it.

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$HG = \begin{bmatrix} 1 \end{bmatrix}$$

Example

Let

$$G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

- Does GH exist? If so, compute it.
- Does HG exist? If so, compute it.

$$GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$$

$$HG = \begin{bmatrix} 1 \end{bmatrix}$$

In this example, GH and HG both exist, but they are not equal. They aren't even the same size!

Example

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

Example

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

Example

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix}$$

Example

Let

$$P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \text{ and } Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix}$$

- Does PQ exist? If so, compute it.
- Does QP exist? If so, compute it.

$$PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix}$$

$$QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix}$$

In this example, PQ and QP both exist and are the same size, but $PQ \neq QP$.

Fact

The four previous examples illustrate an important property of matrix multiplication.

> *In general, matrix multiplication is not commutative, i.e., the order of the matrices in the product is important.*

Commuting Matrices

Example

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

Commuting Matrices

Example

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Commuting Matrices

Example

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$VU = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

Commuting Matrices

Example

Let

$$U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \text{ and } V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

- Does UV exist? If so, compute it.
- Does VU exist? If so, compute it.

$$UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

$$VU = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

In this particular example, the matrices commute, i.e., $UV = VU$.

Summary

1. More Matrix Transformations
2. Matrix Multiplication
3. Commutativity of Matrix Multiplication