Recall

1. Associated Homogeneous Systems
2. Matrix Transformations
Today

1. More Matrix Transformations

2. Matrix Multiplication

3. Commutativity of Matrix Multiplication
Zero and One

1. If $A$ is the $m \times n$ matrix of all zeros, then the transformation induced by $A$, namely

   \[ T(x) = 0 \text{ for all } x \in \mathbb{R}^n, \]

   is called the zero transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$, and is written $T = 0$.

2. If $A$ is the $n \times n$ identity matrix, then the transformation induced by $A$ is called the identity transformation on $\mathbb{R}^n$, and is written $1_{\mathbb{R}^n}$. We have

   \[ 1_{\mathbb{R}^n}(x) = x \text{ for all } x \in \mathbb{R}^n. \]
Recall that we can specify a vector \[ \begin{bmatrix} a \\ b \end{bmatrix} \in \mathbb{R}^2 \] either in rectangular coordinates or in polar coordinates. In polar coordinates, we specify the distance \( r \) from the origin and the angle \( \theta \) from the positive \( x \)-axis. The two systems of coordinates are related by

\[
\begin{align*}
    a &= r \cos(\theta) \\
    b &= r \sin(\theta)
\end{align*}
\]

\[
\begin{align*}
    r &= \sqrt{a^2 + b^2} \\
    \theta &= \begin{cases} 
        \tan^{-1}\left(\frac{b}{a}\right) & \text{if } a > 0 \\
        \pi + \tan^{-1}\left(\frac{b}{a}\right) & \text{if } a < 0
    \end{cases}
\end{align*}
\]
Rotations

Suppose we want to rotate counterclockwise by an angle $\alpha$. Then a point with polar coordinates $(r, \theta)$ will map to one with polar coordinates $(r, \theta + \alpha)$. In rectangular coordinates, this means

$$\begin{bmatrix} r \cos(\theta) \\ r \sin(\theta) \end{bmatrix} \rightarrow \begin{bmatrix} r \cos(\theta + \alpha) \\ r \sin(\theta + \alpha) \end{bmatrix}$$

$$= \begin{bmatrix} r \cos(\theta) \cos(\alpha) - r \sin(\theta) \sin(\alpha) \\ r \sin(\theta) \cos(\alpha) + r \cos(\theta) \sin(\alpha) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \begin{bmatrix} r \cos(\theta) \\ r \sin(\theta) \end{bmatrix},$$

so rotation by $\alpha$ is a matrix transformation, induced by the matrix

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}.$$
A matrix transformation must map the zero vector \( \mathbf{0} \) to itself.

**Example**

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be defined by

\[
T(\mathbf{x}) = \mathbf{x} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}
\]

for all \( \mathbf{x} \in \mathbb{R}^2 \).

This translation is not a matrix transformation.
Matrix Multiplication

Let $A$ be an $m \times n$ matrix and $B = [b_1, b_2, \ldots, b_k]$ an $n \times k$ matrix, whose columns are $b_1, b_2, \ldots, b_k$. The product of $A$ and $B$ is the matrix

$$AB = A [b_1, b_2, \ldots, b_k] = [Ab_1, Ab_2, \ldots, Ab_k],$$

i.e., the first column of $AB$ is $Ab_1$, the second column of $AB$ is $Ab_2$, etc.

Note that the number of **columns** of the first matrix must match the number of **rows** of the second.
Example

Let $A$ and $B$ be matrices,

$$A = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} -1 & 1 & 2 \\ 0 & -2 & 4 \\ 1 & 0 & 0 \end{bmatrix}$$

Then $AB$ has columns

$$Ab_1 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix},$$

$$Ab_2 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \end{bmatrix},$$

$$Ab_3 = \begin{bmatrix} -1 & 0 & 3 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$$
Example (continued)

Putting the columns together, we get

\[
\begin{bmatrix}
-1 & 0 & 3 \\
2 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 2 \\
0 & -2 & 4 \\
1 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
4 & -1 & -2 \\
-1 & 4 & 0
\end{bmatrix}.
\]
Theorem (§2.3 Theorem 1)

Let $A$ be an $m \times n$ matrix, and $B$ an $n \times k$ matrix. Then

$$A(Bx) = (AB)x \text{ for all } k\text{-vectors } x \in \mathbb{R}^k.$$  

Proof.

We can write $Bx$ as $x_1b_1 + \cdots + x_kb_k$, so

$$A(Bx) = x_1Ab_1 + \cdots + x_kAb_k.$$  

On the other hand, the columns of $AB$ are $Ab_1, \ldots, Ab_k$, so we have

$$(AB)x = x_1Ab_1 + \cdots + x_kAb_k.$$
Suppose that $\mathbb{R}^k \xrightarrow{T_B} \mathbb{R}^m \xrightarrow{T_A} \mathbb{R}^n$ are matrix transformations, induced by an $n \times m$ matrix $A$ and an $m \times k$ matrix $B$. The composite of $T_A$ and $T_B$, written $T_A \circ T_B$, is defined by $(T_A \circ T_B)(x) = T_A(T_B(x))$ for $x \in \mathbb{R}^k$.

The theorem implies that the composite of two matrix transformations is a matrix transformation, induced by $AB$.

Any invertible matrix transformation of $\mathbb{R}^2$ can be written as a composite of shears, reflections and $x$ or $y$-expansions.
Dot Products and Matrix Multiplication

Theorem (§2.3 Theorem 2)

Let $A$ be an $m \times n$ matrix and $B$ and $n \times k$ matrix. Then the $(i, j)$-entry of $AB$ is the dot product of row $i$ of $A$ with column $j$ of $B$. 

Example

Use the above theorem to compute

$$
\begin{bmatrix}
-1 & 0 & 3 \\
-2 & 1 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
-1 & 1 & 2 \\
0 & -2 & 4 \\
1 & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
4 \\
-1 \\
-2
\end{bmatrix}
$$
Dot Products and Matrix Multiplication

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\end{bmatrix}
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Dot Products and Matrix Multiplication

**Theorem (§2.3 Theorem 2)**

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1 & 0 & 0
\end{bmatrix}
= 
\begin{bmatrix}
4 & -1 & -2 \\
-1 & 4 & 0
\end{bmatrix}
$$
Compatible Sizes

- Let $A$ and $B$ be matrices. In order for the product $AB$ to exist, the number of rows in $B$ must be equal to the number of columns in $A$.

- Assuming that $A$ is an $m \times n$ matrix, the product $AB$ is defined if and only if $B$ is an $n \times k$ matrix for some $k$. If the product is defined, then $A$ and $B$ are said to be compatible for (matrix) multiplication.

- Given that $A$ is $m \times n$ and $B$ is $n \times k$, the product $AB$ is an $m \times k$ matrix.
Example (revisited)

As we saw earlier,

\[
\begin{bmatrix}
2 & 0 & 3 \\
3 & -1 & 1
\end{bmatrix}
\begin{bmatrix}
-1 & 1 & 2 \\
0 & -2 & 4 \\
1 & 0 & 0
\end{bmatrix}
= \begin{bmatrix}
4 & -1 & -2 \\
-1 & 4 & 0
\end{bmatrix}
\]
Example (revisited)

As we saw earlier,

\[
\begin{pmatrix}
  -1 & 0 & 3 \\ 
  2 & -1 & 1 \\
\end{pmatrix}^{2 \times 3} \begin{pmatrix}
  -1 & 1 & 2 \\ 
  0 & -2 & 4 \\ 
  1 & 0 & 0 \\
\end{pmatrix}^{3 \times 3} = \begin{pmatrix}
  4 & -1 & -2 \\
\end{pmatrix}^{2 \times 3}
\]

Note that the product

\[
\begin{pmatrix}
  -1 & 1 & 2 \\ 
  0 & -2 & 4 \\ 
  1 & 0 & 0 \\
\end{pmatrix}^{3 \times 3} \begin{pmatrix}
  -1 & 0 & 3 \\
  2 & -1 & 1 \\
\end{pmatrix}^{2 \times 3}
\]

does not exist.
Example

Let

\[
A = \begin{bmatrix}
1 & 2 \\
-3 & 0 \\
1 & -4
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 & -1 & 2 & 0 \\
3 & -2 & 1 & -3
\end{bmatrix}
\]

- Does \( AB \) exist? If so, compute it.
- Does \( BA \) exist? If so, compute it.
Example

Let

\[
A = \begin{bmatrix}
1 & 2 \\
-3 & 0 \\
1 & -4
\end{bmatrix}
\quad \text{and} \quad
B = \begin{bmatrix}
1 & -1 & 2 & 0 \\
3 & -2 & 1 & -3
\end{bmatrix}
\]

\begin{itemize}
  \item Does \( AB \) exist? If so, compute it.
  \item Does \( BA \) exist? If so, compute it.
\end{itemize}

\[
AB = \begin{bmatrix}
7 & -5 & 4 & -6 \\
-3 & 3 & -6 & 0 \\
-11 & 7 & -2 & 12
\end{bmatrix}
\]

\( BA \) does not exist
Example

Let

\[ A = \begin{bmatrix}
  1 & 2 \\
  -3 & 0 \\
  1 & -4
\end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix}
  1 & -1 & 2 & 0 \\
  3 & -2 & 1 & -3
\end{bmatrix} \]

- Does \( AB \) exist? If so, compute it.
- Does \( BA \) exist? If so, compute it.

\[ AB = \begin{bmatrix}
  7 & -5 & 4 & -6 \\
  -3 & 3 & -6 & 0 \\
  -11 & 7 & -2 & 12
\end{bmatrix} \]

\( BA \) does not exist
Example

Let

\[ G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

- Does \( GH \) exist? If so, compute it.
- Does \( HG \) exist? If so, compute it.
Example

Let

\[ G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ and } H = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

- Does \( GH \) exist? If so, compute it.
- Does \( HG \) exist? If so, compute it.

\[ GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \]

In this example, \( GH \) and \( HG \) both exist, but they are not equal.

They aren't even the same size!
Example

Let

\[ G = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad H = \begin{bmatrix} 1 & 0 \end{bmatrix} \]

- Does \( GH \) exist? If so, compute it.
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Example

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- Does \( GH \) exist? If so, compute it.
- Does \( HG \) exist? If so, compute it.

\[ GH = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \]

\[ HG = \begin{bmatrix} 1 \end{bmatrix} \]

In this example, \( GH \) and \( HG \) both exist, but they are not equal. They aren’t even the same size!
Example

Let

\[ P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \]

- Does \( PQ \) exist? If so, compute it.
- Does \( QP \) exist? If so, compute it.
Example

Let

\[ P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \]

- Does \( PQ \) exist? If so, compute it.
- Does \( QP \) exist? If so, compute it.

\[ PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \]
Example

Let

\[ P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \]

- Does \( PQ \) exist? If so, compute it.
- Does \( QP \) exist? If so, compute it.

\[ PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \]

\[ QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix} \]
Example

Let

\[ P = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix} \quad \text{and} \quad Q = \begin{bmatrix} -1 & 1 \\ 0 & 3 \end{bmatrix} \]

- Does \( PQ \) exist? If so, compute it.
- Does \( QP \) exist? If so, compute it.

\[ PQ = \begin{bmatrix} -1 & 1 \\ -2 & -1 \end{bmatrix} \]

\[ QP = \begin{bmatrix} 1 & -1 \\ 6 & -3 \end{bmatrix} \]

In this example, \( PQ \) and \( QP \) both exist and are the same size, but \( PQ \neq QP \).
The four previous examples illustrate an important property of matrix multiplication.

In general, matrix multiplication is not commutative, i.e., the order of the matrices in the product is important.
Commuting Matrices

Example

Let

\[ U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

- Does \( UV \) exist? If so, compute it.
- Does \( VU \) exist? If so, compute it.
Commuting Matrices

Example

Let

\[ U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

- Does \( UV \) exist? If so, compute it.
- Does \( VU \) exist? If so, compute it.

\[ UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \]
Commuting Matrices

**Example**

Let

\[ U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

- Does \( UV \) exist? If so, compute it.
- Does \( VU \) exist? If so, compute it.

\[ UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \]

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Commuting Matrices

Example

Let

\[ U = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \]

- Does \( UV \) exist? If so, compute it.
- Does \( VU \) exist? If so, compute it.

\[ UV = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \]

\[ VU = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix} \]

In this particular example, the matrices commute, i.e., \( UV = VU \).
Summary

1. More Matrix Transformations
2. Matrix Multiplication
3. Commutativity of Matrix Multiplication