Recall

1. Vectors
2. The Matrix-Vector Product
3. The Dot Product
Today

1. Associated Homogeneous Systems

2. Matrix Transformations
Given a linear system \( Ax = b \), the system \( Ax = 0 \) is called the associated homogeneous system.

**Theorem (§2.2 Theorem 3)**

Suppose that \( x_1 \) is a particular solution to the system of linear equations \( Ax = b \).

- If \( x_0 \) is a solution to the associated homogeneous system then \( x_1 + x_0 \) is another solution to \( Ax = b \).
- Every solution to \( Ax = b \) has the form \( x_1 + x_0 \) for some solution \( x_0 \) to the associated homogeneous system.

Namely, we can go back and forth between solutions to \( Ax = b \) and solutions to \( Ax = 0 \) by adding or subtracting some solution \( x_1 \).
Example

The system of linear equations $A\mathbf{x} = \mathbf{b}$, with

$$
A = \begin{bmatrix}
2 & 1 & 3 & 3 \\
0 & 1 & -1 & 1 \\
-1 & 1 & -3 & 0
\end{bmatrix}
$$

and

$$
\mathbf{b} = \begin{bmatrix}
4 \\
2 \\
1
\end{bmatrix}
$$

has solution

$$
\mathbf{x} = \begin{bmatrix}
1 - 2s - t \\
2 + s - t \\
s \\
t
\end{bmatrix}
= \begin{bmatrix}
1 \\
2 \\
0 \\
0
\end{bmatrix}
+ s \begin{bmatrix}
-2 \\
1 \\
1 \\
0
\end{bmatrix}
+ t \begin{bmatrix}
-1 \\
-1 \\
0 \\
1
\end{bmatrix}, \quad s, t \in \mathbb{R}.
$$
Example (continued)

Furthermore, \( x_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \) is a particular solution to \( Ax = b \)

(obtained by setting \( s = t = 0 \)), while

\[
x_0 = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}
\]

is the general solution, in parametric form, to the associated homogeneous system \( Ax = 0 \).

**Example 7** (p. 48) is similar.
Matrix Transformations

Examples

- In $\mathbb{R}^2$, reflection in the $x$-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$.
- In $\mathbb{R}^2$, reflection in the $y$-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$.

A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ (also called just a transformation or function) is a rule assigning a vector in $\mathbb{R}^m$ to each vector in $\mathbb{R}^n$. We write

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ or } \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m.$$ 

If $m = n$, then we say $T$ is a transformation of $\mathbb{R}^n$. 
Specifying a transformation symbolically

**Example**

\[ T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \text{ defined by} \]

\[
T \begin{bmatrix}
  a \\
  b \\
  c
\end{bmatrix} = \begin{bmatrix}
  a + b \\
  b + c \\
  a - c \\
  c - b
\end{bmatrix}
\]

is a transformation.
Specifying a transformation pictorially

Reflection in the $x$-axis: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ -b \end{bmatrix}$. 
Specifying a transformation pictorially

Reflection in the $y$-axis: \[
\begin{bmatrix}
a \\
b
\end{bmatrix} \mapsto \begin{bmatrix}
-a \\
b
\end{bmatrix}.
\]
Specifying a transformation pictorially

Rotation by $\frac{\pi}{2}$: \[
\begin{bmatrix}
a \\
b
\end{bmatrix} \mapsto \begin{bmatrix}
-b \\
a
\end{bmatrix}.
\]
Specifying a transformation pictorially

Expansion by a factor of 2: \[
\begin{bmatrix}
a \\
b
\end{bmatrix} \mapsto \begin{bmatrix}
2a \\
2b
\end{bmatrix}.
\]
Specifying a transformation pictorially

Compression by a factor of 2: \[
\begin{bmatrix}
a \\
b
\end{bmatrix} \mapsto \begin{bmatrix}
a/2 \\
b/2
\end{bmatrix}.
\]
Specifying a transformation pictorially

\[ \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 2a \\ b \end{bmatrix}. \]
Specifying a transformation pictorially

\[ \begin{bmatrix} a \\ b \end{bmatrix} \rightarrow \begin{bmatrix} a/2 \\ b \end{bmatrix}. \]
Specifying a transformation pictorially

Positive $x$-shear: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a + b/2 \\ b \end{bmatrix}$. 
Specifying a transformation pictorially

Negative $x$-shear: \[
\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a - b/2 \\ b \end{bmatrix}.
\]
To specify a transformation pictorially, we need to know that the whole picture extends what we can see. It suffices to know that the transformation is linear:

1. \( T(x + y) = T(x) + T(y) \) for all \( x, y \in \mathbb{R}^n \),
2. \( T(kx) = kT(x) \) for all \( x \in \mathbb{R}^n \) and \( k \in \mathbb{R} \).

We will return to this notion in §2.6.
Matrix Transformation

Let $A$ be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(x) = Ax \text{ for each } x \in \mathbb{R}^n$$

is called the matrix transformation induced by $A$. 
Matrix Transformation

Let $A$ be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \rightarrow \mathbb{R}^m$ defined by

$$T(x) = Ax$$

for each $x \in \mathbb{R}^n$ is called the matrix transformation induced by $A$.

Example

In $\mathbb{R}^2$, reflection in the $x$-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$
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**Example**

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Example

The transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) defined by

\[
T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix}
\]

is a matrix transformation.
Example

The transformation \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^4 \) defined by

\[
T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a + b \\ b + c \\ a - c \\ c - b \end{bmatrix}
\]

is a matrix transformation.

\( T \) is induced by the matrix

\[
A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}
\]
Definition (Equality of Transformations)

Suppose \( S : \mathbb{R}^n \to \mathbb{R}^m \) and \( T : \mathbb{R}^n \to \mathbb{R}^m \) are transformations. Then \( S = T \) if and only if \( S(x) = T(x) \) for every \( x \in \mathbb{R}^n \).
Summary

1. Associated Homogeneous Systems

2. Matrix Transformations