Linear Methods (Math 211) - Lecture 6, §2.2

(with slides adapted from K. Seyffarth)

David Roe

September 20, 2013

Recall

1. Vectors
2. The Matrix-Vector Product
3. The Dot Product

Today

1. Associated Homogeneous Systems

2. Matrix Transformations

Given a linear system $A\mathbf{x} = \mathbf{b}$, the system $A\mathbf{x} = \mathbf{0}$ is called the associated homogeneous system.

Theorem (§2.2 Theorem 3)

Suppose that $\mathbf{x}_1$ is a particular solution to the system of linear equations $A\mathbf{x} = \mathbf{b}$.

- *If $\mathbf{x}_0$ is a solution to the associated homogeneous system then $\mathbf{x}_1 + \mathbf{x}_0$ is another solution to $A\mathbf{x} = \mathbf{b}$.*
- *Every solution to $A\mathbf{x} = \mathbf{b}$ has the form $\mathbf{x}_1 + \mathbf{x}_0$ for some solution $\mathbf{x}_0$ to the associated homogeneous system.*

Namely, we can go back and forth between solutions to $A\mathbf{x} = \mathbf{b}$ and solutions to $A\mathbf{x} = 0$ by adding or subtracting some solution x_1.

Example

The system of linear equations $A\mathbf{x} = \mathbf{b}$, with

$$A = \begin{bmatrix} 2 & 1 & 3 & 3 \\ 0 & 1 & -1 & 1 \\ -1 & 1 & -3 & 0 \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

has solution

$$\mathbf{x} = \begin{bmatrix} 1 - 2s - t \\ 2 + s - t \\ s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

Example (continued)

Furthermore, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ is a particular solution to $A\mathbf{x} = \mathbf{b}$ (obtained by setting $s = t = 0$), while

$$\mathbf{x}_0 = s \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad s, t \in \mathbb{R}$$

is the general solution, in parametric form, to the associated homogeneous system $A\mathbf{x} = \mathbf{0}$.

Example 7 (p. 48) is similar.

Matrix Transformations

Examples

- In $\mathbb{R}^2$, reflection in the x-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$.
- In $\mathbb{R}^2$, reflection in the y-axis transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} -a \\ b \end{bmatrix}$.

A transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ (also called just a transformation or function) is a rule assigning a vector in $\mathbb{R}^m$ to each vector in $\mathbb{R}^n$. We write

$$T : \mathbb{R}^n \to \mathbb{R}^m \text{ or}$$

$$\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m.$$

If $m = n$, then we say T is a transformation of $\mathbb{R}^n$.

Specifying a transformation symbolically

Example

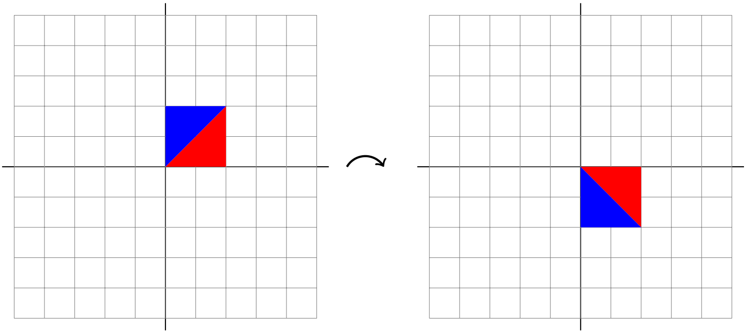
$T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

is a transformation.

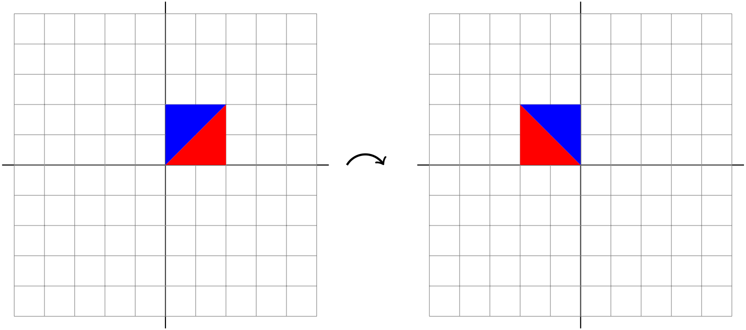
Specifying a transformation pictorially

Reflection in the x-axis: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a \\ -b \end{bmatrix}$.



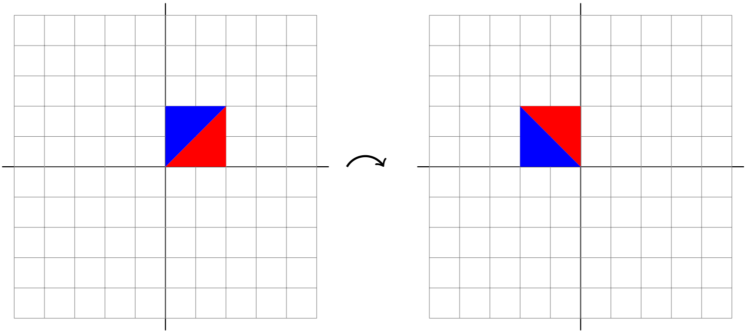
Specifying a transformation pictorially

Reflection in the y-axis: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} -a \\ b \end{bmatrix}$.



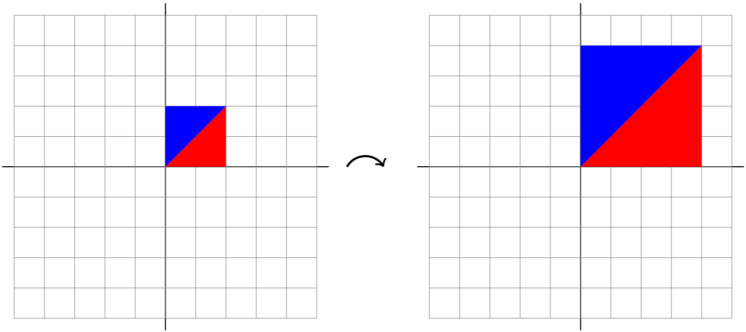
Specifying a transformation pictorially

Rotation by $\frac{\pi}{2}$: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} -b \\ a \end{bmatrix}$.



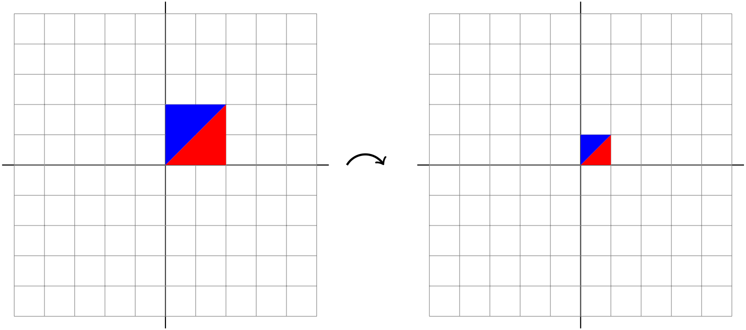
Specifying a transformation pictorially

Expansion by a factor of 2: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 2a \\ 2b \end{bmatrix}$.



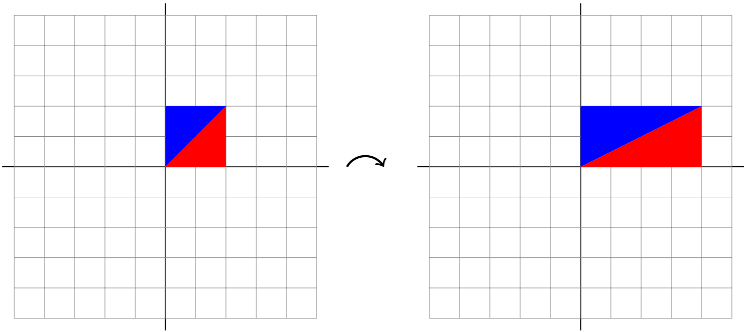
Specifying a transformation pictorially

Compression by a factor of 2: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a/2 \\ b/2 \end{bmatrix}.$



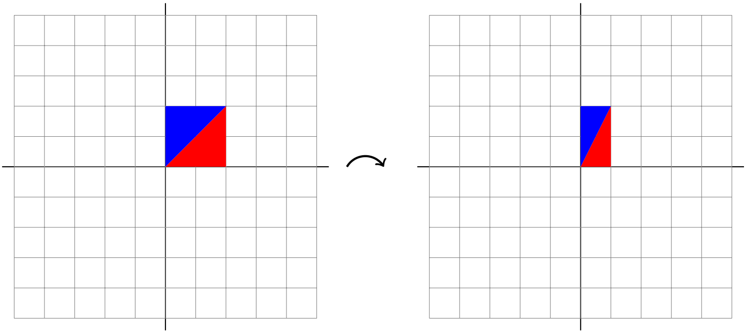
Specifying a transformation pictorially

$$x\text{-expansion: } \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} 2a \\ b \end{bmatrix}.$$



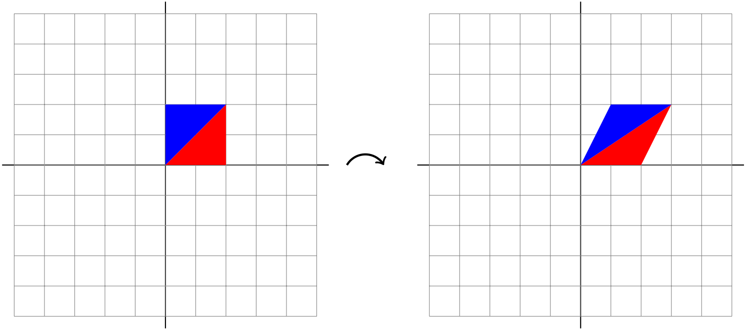
Specifying a transformation pictorially

$$x\text{-contraction: } \begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a/2 \\ b \end{bmatrix}.$$



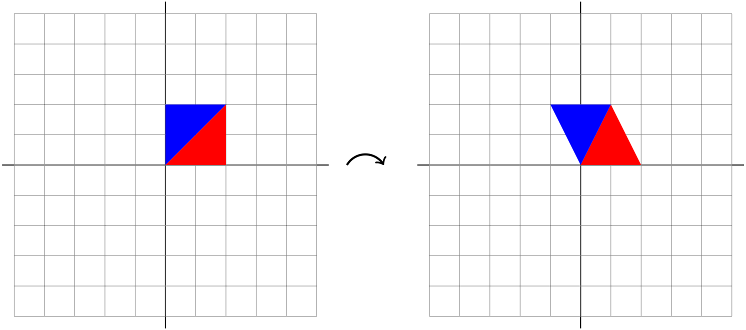
Specifying a transformation pictorially

Positive x-shear: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a + b/2 \\ b \end{bmatrix}.$



Specifying a transformation pictorially

Negative x-shear: $\begin{bmatrix} a \\ b \end{bmatrix} \mapsto \begin{bmatrix} a - b/2 \\ b \end{bmatrix}.$



Linear Transformations

To specify a transformation pictorially, we need to know that that the whole picture extends what we can see. It suffices to know that the transformation is linear:

1. $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$,
2. $T(k\mathbf{x}) = kT(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$ and $k \in \mathbb{R}$.

We will return to this notion in §2.6.

Matrix Transformation

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x} \text{ for each } \mathbf{x} \in \mathbb{R}^n$$

is called the matrix transformation induced by A.

Matrix Transformation

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x} \text{ for each } \mathbf{x} \in \mathbb{R}^n$$

is called the matrix transformation induced by A.

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Matrix Transformation

Let A be an $m \times n$ matrix. The transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\mathbf{x}) = A\mathbf{x} \text{ for each } \mathbf{x} \in \mathbb{R}^n$$

is called the matrix transformation induced by A.

Example

In $\mathbb{R}^2$, reflection in the x-axis, which transforms $\begin{bmatrix} a \\ b \end{bmatrix}$ to $\begin{bmatrix} a \\ -b \end{bmatrix}$, is a matrix transformation because

$$\begin{bmatrix} a \\ -b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}.$$

Preview: linear transformations and matrix transformations are the same class of transformations.

Example

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

is a matrix transformation.

Example

The transformation $T : \mathbb{R}^3 \to \mathbb{R}^4$ defined by

$$T \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a+b \\ b+c \\ a-c \\ c-b \end{bmatrix}$$

is a matrix transformation.

T is induced by the matrix

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{bmatrix}$$

Definition (Equality of Transformations)

Suppose $S : \mathbb{R}^n \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^m$ are transformations. Then $S = T$ if and only if $S(\mathbf{x}) = T(\mathbf{x})$ for every $\mathbf{x} \in \mathbb{R}^n$.

Summary

1. Associated Homogeneous Systems
2. Matrix Transformations