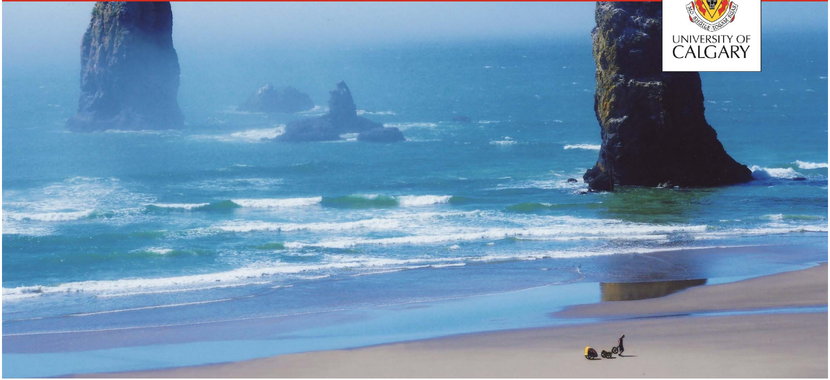
**CENTRE FOR INTERNATIONAL STUDENTS AND STUDY ABROAD**

UNIVERSITY OF CALGARY



# Find out how to Study Abroad!

Wednesdays @ 1:00pm & Thursdays @ 11:00am
45 minute info sessions in CISSA (MSC 275)

@uccissa ucalgary.ca/uci/abroad

# Linear Methods (Math 211) - Lecture 5, §2.2

(with slides adapted from K. Seyffarth)

David Roe

September 18, 2013

# Recall

1. Matrices
2. Matrix Addition and Scalar Multiplication
3. Transposition and Symmetric Matrices
4. Examples

# Today

1. Vectors
2. The Matrix-Vector Product
3. The Dot Product

## Example

The linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 2 \\ -x_1 & + & 4x_2 & + & 5x_3 & - & 2x_4 & = & 1 \\ x_1 & + & 6x_2 & + & 3x_3 & + & 4x_4 & = & -1 \end{array}$$

has **coefficient** matrix $A$ and **constant** matrix $B$, where

$$A = \begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

## Example

The linear system

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & + & 3x_4 & = & 2 \\ -x_1 & + & 4x_2 & + & 5x_3 & - & 2x_4 & = & 1 \\ x_1 & + & 6x_2 & + & 3x_3 & + & 4x_4 & = & -1 \end{array}$$

has **coefficient** matrix $A$ and **constant** matrix $B$, where

$$A = \begin{bmatrix} 1 & 1 & -1 & 3 \\ -1 & 4 & 5 & -2 \\ 1 & 6 & 3 & 4 \end{bmatrix} \text{ and } B = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Using (matrix) addition and scalar multiplication, we can rewrite this system as

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 4 \\ 6 \end{bmatrix} x_2 + \begin{bmatrix} -1 \\ 5 \\ -2 \end{bmatrix} x_3 + \begin{bmatrix} 3 \\ -2 \\ 4 \end{bmatrix} x_4 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

This example illustrates the fact that solving a system of linear equations is equivalent to finding the coefficients of a linear combination of the columns of the coefficient matrix $A$ so that the result is equal to the constant matrix $B$.

# Notation and Terminology

- $\mathbb{R}$: the set of real numbers.
- $\mathbb{R}^n$: set of columns (with entries from $\mathbb{R}$) having $n$ rows.

$$\begin{bmatrix} 1 \\ -1 \\ 0 \\ 3 \end{bmatrix} \in \mathbb{R}^4, \begin{bmatrix} -6 \\ 5 \end{bmatrix} \in \mathbb{R}^2, \begin{bmatrix} 2 \\ 3 \\ -7 \end{bmatrix} \in \mathbb{R}^3.$$

- The columns of $\mathbb{R}^n$ are also called vectors or $n$-vectors.
- To save space, a vector is sometimes written as the transpose of a row matrix.

$$\begin{bmatrix} 1 & -1 & 0 & 3 \end{bmatrix}^T \in \mathbb{R}^4$$

# The Matrix-Vector Product

Let $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$ be an $m \times n$ matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$, and $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \ldots & x_n \end{bmatrix}^T$ any $n$-vector.
The product $A\mathbf{x}$ is defined as the $m$-vector given by

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots \mathbf{a}_n x_n,$$

i.e., $A\mathbf{x}$ is a **linear combination** of the columns of $A$ (and the coefficients are the entries of $\mathbf{x}$, in order).
As with matrix addition, there is a constraint on the size of the inputs: the number of **columns** of $A$ must equal the number of **rows** of $\mathbf{x}$.

# Matrix Equations

If a system of $m$ linear equations in $n$ variables has the $m \times n$ matrix $A$ as its coefficient matrix, the $n$-vector $\mathbf{b}$ as its constant matrix, and the $n$-vector $\mathbf{x}$ as the matrix of variables, then the system can be written as the matrix equation

$$A\mathbf{x} = \mathbf{b}.$$

**Theorem (§2.2 Theorem 1)**

- *Every system of linear equations has the form* $A\mathbf{x} = \mathbf{b}$ *where* $A$ *is the coefficient matrix,* $\mathbf{x}$ *is the matrix of variables, and* $\mathbf{b}$ *is the constant matrix.*
- $A\mathbf{x} = \mathbf{b}$ *is* **consistent** *if and only if* $\mathbf{b}$ *is a linear combination of the columns of* $A$.
- *If* $A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \dots & \mathbf{a}_n \end{bmatrix}$, *then* $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \dots & x_n \end{bmatrix}^T$ *is a solution to* $A\mathbf{x} = \mathbf{b}$ *if and only if* $x_1, x_2, \dots, x_n$ *are a solution to the vector equation*

$$\mathbf{a}_1 x_1 + \mathbf{a}_2 x_2 + \cdots \mathbf{a}_n x_n = \mathbf{b}.$$

## Example

Let

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix}$$

1. Compute $A\mathbf{y}$.
2. Can $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ be expressed as a linear combination of the columns of $A$? If so, find a linear combination that does so.

## Example (continued)

1. $$A\mathbf{y} = 2\begin{bmatrix}1\\2\\3\end{bmatrix} + (-1)\begin{bmatrix}0\\-1\\1\end{bmatrix} + 1\begin{bmatrix}2\\0\\3\end{bmatrix} + 4\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\9\\12\end{bmatrix}$$

## Example (continued)

1. $$A\mathbf{y} = 2\begin{bmatrix}1\\2\\3\end{bmatrix} + (-1)\begin{bmatrix}0\\-1\\1\end{bmatrix} + 1\begin{bmatrix}2\\0\\3\end{bmatrix} + 4\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\9\\12\end{bmatrix}$$

2. Solve the system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = \begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T$. To do this, put the **augmented matrix** $\left[A \mid \mathbf{b}\right]$ in reduced row-echelon form.

$$\left[\begin{array}{rrrr|r}1 & 0 & 2 & -1 & 1\\2 & -1 & 0 & 1 & 1\\3 & 1 & 3 & 1 & 1\end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r}1 & 0 & 0 & 1 & \frac{1}{7}\\0 & 1 & 0 & 1 & -\frac{5}{7}\\0 & 0 & 1 & -1 & \frac{3}{7}\end{array}\right]$$

## Example (continued)

1. $A\mathbf{y} = 2\begin{bmatrix}1\\2\\3\end{bmatrix} + (-1)\begin{bmatrix}0\\-1\\1\end{bmatrix} + 1\begin{bmatrix}2\\0\\3\end{bmatrix} + 4\begin{bmatrix}-1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\9\\12\end{bmatrix}$

2. Solve the system $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = \begin{bmatrix}x_1 & x_2 & x_3 & x_4\end{bmatrix}^T$. To do this, put the **augmented matrix** $\left[A \mid \mathbf{b}\right]$ in reduced row-echelon form.

$$\left[\begin{array}{rrrr|r}1 & 0 & 2 & -1 & 1\\ 2 & -1 & 0 & 1 & 1\\ 3 & 1 & 3 & 1 & 1\end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r}1 & 0 & 0 & 1 & \frac{1}{7}\\ 0 & 1 & 0 & 1 & -\frac{5}{7}\\ 0 & 0 & 1 & -1 & \frac{3}{7}\end{array}\right]$$

Since there are infinitely many solutions, simply choose a value for $x_4$. Taking $x_4 = 0$ gives us

$$\begin{bmatrix}1\\1\\1\end{bmatrix} = \frac{1}{7}\begin{bmatrix}1\\2\\3\end{bmatrix} - \frac{5}{7}\begin{bmatrix}0\\-1\\1\end{bmatrix} + \frac{3}{7}\begin{bmatrix}2\\0\\3\end{bmatrix}.$$

**Example (Example 5, p. 44.)**

Write $\mathbf{0}$ for the $m$-vector of all zeros.

- If $A$ is the $m \times n$ matrix of all zeros, then $A\mathbf{x} = \mathbf{0}$ for any $n$-vector $\mathbf{x}$.
- If $\mathbf{x}$ is the $n$-vector of zeros, then $A\mathbf{x} = \mathbf{0}$ for any $m \times n$ matrix $A$.

As with matrices, we will generally use the symbol $\mathbf{0}$ to refer to a zero vector of any size.

# Properties of Matrix-Vector Multiplication

**Theorem (§2.2 Theorem 2)**

*Let $A$ and $B$ be $m \times n$ matrices, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ be $n$-vectors, and $k \in \mathbb{R}$ be a scalar.*

1. $A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}$
2. $A(k\mathbf{x}) = k(A\mathbf{x}) = (kA)\mathbf{x}$
3. $(A + B)\mathbf{x} = A\mathbf{x} + B\mathbf{x}$

# The Dot Product

The dot product of two $n$-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ is the number (scalar)

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

# The Dot Product

The dot product of two $n$-tuples $(a_1, a_2, \ldots, a_n)$ and $(b_1, b_2, \ldots, b_n)$ is the number (scalar)

$$a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

**Theorem (§2.2 Theorem 4)**

*Suppose that $A$ is an $m \times n$ matrix and that $\mathbf{x}$ is an $n$-vector. Then the $i^{th}$ entry of $A\mathbf{x}$ is the dot product of the $i^{th}$ row of $A$ with $\mathbf{x}$.*

## Example

Compute the product

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

using dot products.

## Example

Compute the product

$$\begin{bmatrix} 1 & 0 & 2 & -1 \\ 2 & -1 & 0 & 1 \\ 3 & 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 0 \\ 9 \\ 12 \end{bmatrix}$$

using dot products.

$$\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} = 0 \qquad \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} = 9 \qquad \begin{bmatrix} 3 \\ 1 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \\ 4 \end{bmatrix} = 12$$

The $n \times n$ identity matrix, denoted $I_n$ is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

**Example 11** (p. 49) shows that for any $n$-vector $\mathbf{x}$, $I_n\mathbf{x} = \mathbf{x}$.

The $n \times n$ identity matrix, denoted $I_n$ is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

**Example 11** (p. 49) shows that for any $n$-vector $\mathbf{x}$, $I_n\mathbf{x} = \mathbf{x}$.

For each $j$, $1 \leq j \leq n$, we denote by $\mathbf{e}_j$ the $j^{\text{th}}$ column of $I_n$.

The $n \times n$ identity matrix, denoted $I_n$ is the matrix having ones on its main diagonal and zeros elsewhere, and is defined for all $n \geq 2$.

**Example 11** (p. 49) shows that for any $n$-vector $\mathbf{x}$, $I_n\mathbf{x} = \mathbf{x}$.

For each $j$, $1 \leq j \leq n$, we denote by $\mathbf{e}_j$ the $j^{\text{th}}$ column of $I_n$.

**Theorem (§2.2 Theorem 5)**

*Let $A$ and $B$ be $m \times n$ matrices. If $A\mathbf{x} = B\mathbf{x}$ for every $\mathbf{x} \in \mathbb{R}^n$, then $A = B$.*

**Proof.**

$A\mathbf{e}_j = B\mathbf{e}_j$ so the columns of $A$ and $B$ are the same. □

**Problem**

*Find examples of matrices $A$ and $B$, and a vector $\mathbf{x} \neq 0$, so that $A\mathbf{x} = B\mathbf{x}$ but $A \neq B$.*

# Summary

1. Vectors
2. The Matrix-Vector Product
3. The Dot Product