Linear Methods (Math 211) - Lecture 4, §2.1

(with slides adapted from K. Seyffarth)

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September 16, 2013

Recall

Last time:

1. Gaussian elimination
2. Homogeneous systems - linear combinations, qualitative behavior and basic solutions

Today

Matrices - Basic Definitions and Notation

Let m and n be positive integers.

- An $m \times n$ matrix is a rectangular array of numbers having m rows and n columns. Such a matrix is said to have size $m \times n$.

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General notation for an $m \times n$ matrix, A:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} = \left[a_{ij} \right]$$

Matrices - Properties and Operations

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6. **Subtraction:** for $m \times n$ matrices A and B, $A - B = A + (-1)B$.

Matrix form for solutions to linear systems

The reduced row echlon form of the augmented matrix for the system

$$\begin{array}{rcrcrcrcr} x_1 & - & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \\ 2x_1 & - & 4x_2 & + & x_3 & & & = & 5 \\ x_1 & - & 2x_2 & + & 2x_3 & - & 3x_4 & = & 4 \end{array}$$

is

$$\left[\begin{array}{rrrr|r} 1 & -2 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

leading to the solution

$$\begin{aligned} x_1 &= 2 + 2s - t \\ x_2 &= s \\ x_3 &= 1 + 2t \\ x_4 &= t \end{aligned}$$

for parameters s and t.

Matrix form for solutions to linear systems

$$\begin{array}{rcl} x_1 & = & 2+2s-t \\ x_2 & = & s \\ x_3 & = & 1+2t \\ x_4 & = & t \end{array} \quad \text{can be expressed as} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2+2s-t \\ s \\ 1+2t \\ t \end{bmatrix}.$$

But

$$\begin{bmatrix} 2+2s-t \\ s \\ 1+2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Therefore,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

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Matrix Addition and Scalar Multiplication

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Let A, B and C be $m \times n$ matrices, and let k and p be scalars.

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7. $(kp)A = k(pA)$
8. $1A = A$

Matrix Transposition

If A is an $m \times n$ matrix, then its transpose, denoted A^T, is the $n \times m$ matrix whose i^{th} row is the i^{th} column of A, $1 \leq i \leq n$.

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4. $(A + B)^T = A^T + B^T$.

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Examples

$$\begin{bmatrix} 2 & -3 \\ -3 & 17 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 5 \\ 0 & 2 & 11 \\ 5 & 11 & -3 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 5 & -1 \\ 2 & 1 & -3 & 0 \\ 5 & -3 & 2 & -7 \\ -1 & 0 & -7 & 4 \end{bmatrix}$$

are symmetric matrices.

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Compute

$$\begin{bmatrix} 3 & 1 & 8 \\ -2 & 0 & 4 \\ -1 & 1 & -1 \end{bmatrix} + 2 \begin{bmatrix} 1 & 2 & 3 \\ 9 & 0 & -5 \\ 3 & 1 & 0 \end{bmatrix}^T$$

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$$= \begin{bmatrix} 3 & 1 & 8 \\ -2 & 0 & 4 \\ -1 & 1 & -1 \end{bmatrix} + \begin{bmatrix} 2 & 18 & 6 \\ 4 & 0 & 2 \\ 6 & -10 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 19 & 14 \\ 2 & 0 & 6 \\ 5 & -9 & -1 \end{bmatrix}$$

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We have $A = kA^T = k(kA^T)^T = k^2(A^T)^T = k^2 A$ so $(k^2 - 1)A = 0$. So either $k^2 = 1$ or $A = 0$.

Example

Find a condition on a, b, c so that the following system is consistent. When that equation is satisfied, find all solutions.

$$\begin{aligned} x_1 - 3x_2 + 2x_3 &= a \\ 2x_1 - 7x_2 + 4x_3 &= b \\ 4x_1 - 16x_2 + 8x_3 &= c \end{aligned}$$

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$$\to \left[\begin{array}{ccc|r} 1 & 0 & 2 & 7a-3b \\ 0 & 1 & 0 & 2a-b \\ 0 & 0 & 0 & 4a-4b+c \end{array}\right]$$

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So $c = 4b - 4a$. In this case we have $x_3 = t$, $x_1 = 7a - 3b - 2t$ and $x_2 = 2a - b$.

Summary