Linear Methods (Math 211) - Lecture 4, §2.1

(with slides adapted from K. Seyffarth)

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Recall

Last time:

1. Gaussian elimination
2. Homogeneous systems - linear combinations, qualitative behavior and basic solutions
Today

1. Matrices
2. Matrix Addition and Scalar Multiplication
3. Transposition and Symmetric Matrices
4. Examples
Let $m$ and $n$ be positive integers.

- An $m \times n$ matrix is a rectangular array of numbers having $m$ rows and $n$ columns. Such a matrix is said to have size $m \times n$. 

Matrices - Basic Definitions and Notation
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General notation for an $m \times n$ matrix, $A$:

$$A = \begin{bmatrix}
a_{11} & a_{12} & a_{13} & \ldots & a_{1n} \\
a_{21} & a_{22} & a_{23} & \ldots & a_{2n} \\
a_{31} & a_{32} & a_{33} & \ldots & a_{3n} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m1} & a_{m2} & a_{m3} & \ldots & a_{mn}
\end{bmatrix} = [a_{ij}]$$
Equality: two matrices are equal if and only if they have the same size and the corresponding entries are equal.
Matrices - Properties and Operations

1. **Equality**: two matrices are equal if and only if they have the same size and the corresponding entries are equal.

2. **Addition**: matrices must have the same size; add corresponding entries.
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4. **Zero Matrix**: an $m \times n$ matrix with all entries equal to zero.
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5. **Negative of a Matrix**: for an $m \times n$ matrix $A$, its negative is denoted $-A$ and $-A = (-1)A$. 
Matrices - Properties and Operations

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5. **Negative of a Matrix:** for an \( m \times n \) matrix \( A \), its negative is denoted \(-A\) and \(-A = (-1)A\).

6. **Subtraction:** for \( m \times n \) matrices \( A \) and \( B \), \( A - B = A + (-1)B \).
Matrix form for solutions to linear systems

The reduced row echelon form of the augmented matrix for the system

\[
\begin{align*}
  x_1 - 2x_2 - x_3 + 3x_4 &= 1 \\
  2x_1 - 4x_2 + x_3 &= 5 \\
  x_1 - 2x_2 + 2x_3 - 3x_4 &= 4
\end{align*}
\]

is

\[
\begin{bmatrix}
  1 & -2 & 0 & 1 & 2 \\
  0 & 0 & 1 & -2 & 1 \\
  0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

leading to the solution

\[
\begin{align*}
  x_1 &= 2 + 2s - t \\
  x_2 &= s \\
  x_3 &= 1 + 2t \\
  x_4 &= t
\end{align*}
\]

for parameters \( s \) and \( t \).
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\]

can be expressed as

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
2 + 2s - t \\
s \\
1 + 2t \\
t
\end{bmatrix}.
\]

But

\[
\begin{bmatrix}
2 + 2s - t \\
s \\
1 + 2t \\
t
\end{bmatrix} =
\begin{bmatrix}
2 \\
0 \\
1 \\
0
\end{bmatrix}
+ s
\begin{bmatrix}
2 \\
1 \\
0 \\
0
\end{bmatrix}
+ t
\begin{bmatrix}
-1 \\
0 \\
2 \\
1
\end{bmatrix}
\]

Therefore,

\[
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} =
\begin{bmatrix}
2 \\
0 \\
1 \\
0
\end{bmatrix}
+ s
\begin{bmatrix}
2 \\
1 \\
0 \\
0
\end{bmatrix}
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0 \\
2 \\
1
\end{bmatrix}
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for parameters \( s \) and \( t \).
Matrix Addition and Scalar Multiplication

Theorem (§2.1 Theorem 1)

Let $A$, $B$ and $C$ be $m \times n$ matrices, and let $k$ and $p$ be scalars.

1. $A + B = B + A$
Matrix Addition and Scalar Multiplication

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3. There is an $m \times n$ matrix 0 such that $A + 0 = A$ and $0 + A = A$. 
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3. There is an $m \times n$ matrix $0$ such that $A + 0 = A$ and $0 + A = A$.
4. For each $A$ there is an $m \times n$ matrix $-A$ such that $A + (-A) = 0$ and $(-A) + A = 0$. 
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6. $(k + p)A = kA + pA$
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Let \( A, B \) and \( C \) be \( m \times n \) matrices, and let \( k \) and \( p \) be scalars.

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2. \( (A + B) + C = A + (B + C) \)
3. There is an \( m \times n \) matrix 0 such that \( A + 0 = A \) and 0 + \( A = A \).
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5. \( k(A + B) = kA + kB \)
6. \( (k + p)A = kA + pA \)
7. \( (kp)A = k(pA) \)
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6. $(k + p)A = kA + pA$
7. $(kp)A = k(pA)$
8. $1A = A$
Matrix Transposition

If $A$ is an $m \times n$ matrix, then its transpose, denoted $A^T$, is the $n \times m$ matrix whose $i^{th}$ row is the $i^{th}$ column of $A$, $1 \leq i \leq n$. 
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Let $A$ and $B$ be $m \times n$ matrices, and let $k$ be a scalar.

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Let $A$ and $B$ be $m \times n$ matrices, and let $k$ be a scalar.

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2. $(A^T)^T = A.$
Matrix Transposition

If $A$ is an $m \times n$ matrix, then its transpose, denoted $A^T$, is the $n \times m$ matrix whose $i^{\text{th}}$ row is the $i^{\text{th}}$ column of $A$, $1 \leq i \leq n$.

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3. $(kA)^T = kA^T$. 
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1. $A^T$ is an $n \times m$ matrix.
2. $(A^T)^T = A$.
3. $(kA)^T = kA^T$.
4. $(A + B)^T = A^T + B^T$. 


Symmetric Matrices

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the main diagonal of $A$. 
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- Let $A = [a_{ij}]$ be an $m \times n$ matrix. The entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the main diagonal of $A$.
- The matrix $A$ is called symmetric if and only if $A^T = A$. Note that this immediately implies that $A$ is a square matrix.
Symmetric Matrices

Let $A = [a_{ij}]$ be an $m \times n$ matrix. The entries $a_{11}, a_{22}, a_{33}, \ldots$ are called the \textbf{main diagonal} of $A$.

The matrix $A$ is called \textbf{symmetric} if and only if $A^T = A$. Note that this immediately implies that $A$ is a \textbf{square} matrix.

Examples

\[
\begin{bmatrix}
2 & -3 \\
-3 & 17
\end{bmatrix}, \quad \begin{bmatrix}
-1 & 0 & 5 \\
0 & 2 & 11 \\
5 & 11 & -3
\end{bmatrix}, \quad \begin{bmatrix}
0 & 2 & 5 & -1 \\
2 & 1 & -3 & 0 \\
5 & -3 & 2 & -7 \\
-1 & 0 & -7 & 4
\end{bmatrix}
\]

are symmetric matrices.
Example

Compute

\[
\begin{bmatrix}
3 & 1 & 8 \\
-2 & 0 & 4 \\
-1 & 1 & -1 \\
\end{bmatrix}
+ 2
\begin{bmatrix}
1 & 2 & 3 \\
9 & 0 & -5 \\
3 & 1 & 0 \\
\end{bmatrix}^T
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Example

Compute

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3 & 1 & 8 \\
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= \begin{bmatrix}
3 & 1 & 8 \\
-2 & 0 & 4 \\
-1 & 1 & -1
\end{bmatrix}
+ \begin{bmatrix}
2 & 18 & 6 \\
4 & 0 & 2 \\
6 & -10 & 0
\end{bmatrix}
= \begin{bmatrix}
5 & 19 & 14 \\
2 & 0 & 6 \\
5 & -9 & -1
\end{bmatrix}
\]
Example

Show that $A + A^T$ is symmetric for any square matrix $A$. 

The $(i, j)$-entry is $a_{ij} + a_{ji}$, which is the $(j, i)$-entry.
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Example
Suppose $A = kA^T$ for some scalar $k$. Show that either $k = \pm 1$ or $A = 0$. 
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We have $A = kA^T = k(kA^T)^T = k^2(A^T)^T = k^2A$ so $(k^2 - 1)A = 0$. So either $k^2 = 1$ or $A = 0$. 
Example

Find a condition on $a$, $b$, $c$ so that the following system is consistent. When that equation is satisfied, find all solutions.

\[
\begin{align*}
    x_1 - 3x_2 + 2x_3 &= a \\
    2x_1 - 7x_2 + 4x_3 &= b \\
    4x_1 - 16x_2 + 8x_3 &= c
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\[
\begin{bmatrix} 1 & -3 & 2 & | & a \\ 2 & -7 & 4 & | & b \\ 4 & -16 & 8 & | & c \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -3 & 2 & | & a \\ 0 & -1 & 0 & | & b - 2a \\ 0 & -4 & 0 & | & c - 4a \end{bmatrix}
\]

\[
\begin{bmatrix} 1 & 0 & 2 & | & 7a - 3b \\ 0 & 1 & 0 & | & 2a - b \\ 0 & 0 & 0 & | & 4a - 4b + c \end{bmatrix}
\]

So $c = 4b - 4a$. In this case we have $x_3 = t$, $x_1 = 7a - 3b - 2t$ and $x_2 = 2a - b$. 

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Summary

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3. Transposition and Symmetric Matrices
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