Linear Methods (Math 211) - Lecture 3

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September 13, 2013

Recall

Last time:

1. Defined row echelon and reduced row echelon form
2. Defined rank
3. Gaussian elimination - a way to reduce a matrix to echelon form

Today

1. More Gaussian Elimination
2. Homogeneous Systems

$$\begin{bmatrix} {\color{red}0} & 0 & 3 & -15 & 3 \\ -2 & -10 & -9 & 53 & -15 \\ -2 & -10 & -11 & 63 & -16 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 3 & -15 & 3 \\ -2 & -10 & -9 & 53 & -15 \\ -2 & -10 & -11 & 63 & -16 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} -2 & -10 & -9 & 53 & -15 \\ 0 & 0 & 3 & -15 & 3 \\ -2 & -10 & -11 & 63 & -16 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

Swap 1^{st} and 2^{nd} rows

$$\begin{bmatrix} \color{red}{-2} & -10 & -9 & 53 & -15 \\ 0 & 0 & 3 & -15 & 3 \\ -2 & -10 & -11 & 63 & -16 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

$$\begin{bmatrix} -2 & -10 & -9 & 53 & -15 \\ 0 & 0 & 3 & -15 & 3 \\ -2 & -10 & -11 & 63 & -16 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 3 & -15 & 3 \\ -2 & -10 & -11 & 63 & -16 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

Multiply 1^{st} row by $-\frac{1}{2}$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 3 & -15 & 3 \\ -2 & -10 & -11 & 63 & -16 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

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$$\Downarrow$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 3 & -15 & 3 \\ 0 & 0 & -2 & 10 & -1 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

Add 2 times 1^{st} row to 3^{rd} row

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 3 & -15 & 3 \\ 0 & 0 & -2 & 10 & -1 \\ 1 & 5 & 2 & -14 & 5 \end{bmatrix}$$

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Add -1 times 1^{st} row to 4^{th} row

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 3 & -15 & 3 \\ 0 & 0 & -2 & 10 & -1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 3 & -15 & 3 \\ 0 & 0 & -2 & 10 & -1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & -2 & 10 & -1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

Multiply 2^{nd} row by $\frac{1}{3}$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & -2 & 10 & -1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & -2 & 10 & -1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

Add 2 times 2^{nd} row to 3^{rd} row

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{5}{2} & \frac{25}{2} & -\frac{5}{2} \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add $\frac{5}{2}$ times 2^{nd} row to 4^{th} row

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add -1 times 3^{rd} row to 2^{nd} row

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & \frac{15}{2} \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add $-\frac{15}{2}$ times 3^{rd} row to 1^{st} row

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 5 & \frac{9}{2} & -\frac{53}{2} & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & 5 & 0 & -4 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Add $-\frac{9}{2}$ times 2^{nd} row to 1^{st} row

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & -4 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1. What is the rank of the matrix?

$$\left[\begin{array}{rrrr|r} 1 & 5 & 0 & -4 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1. What is the rank of the matrix?
   3, since there are three leading 1s.

$$\left[\begin{array}{cccr|c} 1 & 5 & 0 & -4 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1. What is the rank of the matrix?
   3, since there are three leading 1s.
2. Does the system have any solutions?

$$\left[\begin{array}{cccc|c} 1 & 5 & 0 & -4 & 0 \\ 0 & 0 & 1 & -5 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

1. What is the rank of the matrix?
   3, since there are three leading 1s.
2. Does the system have any solutions?
   No, since one of the rows represents $0 = 1$.

Homogeneous Systems

A linear system is homogeneous if every constant term is zero. For example,

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 0 \\ -x_1 - 5x_2 + x_3 &= 0 \end{aligned}$$

is homogeneous.

Homogeneous Systems

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Any homogenous system has a trivial solution - all variables set to 0.

Linear combinations

Homogeneous equations are special because their solutions have a very nice property.

Suppose $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ are vectors and k is a scalar (i.e. a number). We set

$$\mathbf{x} + \mathbf{y} = \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix} \qquad k\mathbf{x} = \begin{bmatrix} kx_1 \\ kx_2 \\ \vdots \\ kx_n \end{bmatrix}$$

A sum of scalar multiples of several vectors is called a linear combination; $r\mathbf{x} + s\mathbf{y}$ for any scalars r and s for example.

Linear combinations

You can also take linear combinations of more than two vectors. For example,

$$2\begin{bmatrix} 3 \\ 1 \\ -4 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -1 \end{bmatrix}.$$

Solutions to Homogeneous Systems

Suppose that $\mathbf{x}$ and $\mathbf{y}$ are solutions to a homogeneous system. Then so is $r\mathbf{x} + s\mathbf{y}$ for any numbers r and s.
Why? Suppose that $a_1x_1 + \cdots + a_nx_n = 0$ is an equation in the system. Then

$$\begin{aligned}
a_1(rx_1 + sy_1) + \cdots + a_n(rx_n + sy_n) \\
&= ra_1x_1 + sa_1y_1 + \cdots + ra_nx_n + sa_ny_n \\
&= r(a_1x_1 + \cdots a_nx_n) + s(a_1y_1 + \cdots + a_ny_n) \\
&= r \cdot 0 + s \cdot 0 \\
&= 0
\end{aligned}$$

So $r\mathbf{x} + s\mathbf{y}$ is a solution to each equation and thus to the whole system.

Basic solutions

In fact, *every* solution to a homogeneous system can be written uniquely in terms of basic solutions. The basic solutions can be computed using Gaussian elimination; there is one for each parameter.

If the system has n variables and rank r, then there are $n - r$ basic solutions.

Finding basic solutions

The first step in finding basic solutions is Gaussian elimination. Suppose we've done that, and we have the augmented matrix

$$\left[\begin{array}{rrrrrrr|r} 1 & 0 & 6 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$x_3 = r \qquad x_1 = -6r + 3s - t$$

$$x_5 = s \qquad x_2 = 4r + 5s - 2t$$

$$x_7 = t \qquad x_4 = -4s + t$$

$$x_6 = 0.$$

Finding basic solutions

We can rewrite

$$\begin{aligned} x_3 &= r & \qquad x_1 &= -6r + 3s - t \\ x_5 &= s & x_2 &= 4r + 5s - 2t \\ x_7 &= t & x_4 &= -4s + t \\ & & x_6 &= 0 \end{aligned}$$

as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{bmatrix} = r \begin{bmatrix} -6 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 3 \\ 5 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

From the matrix,

$$\left[\begin{array}{ccccccc|c} 1 & 0 & 6 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & -4 & 0 & -5 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & 4 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

we can extract the basic solutions directly:

$$\begin{bmatrix} -6 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 5 \\ 0 \\ -4 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Summary

1. More Gaussian Elimination

2. Homogeneous Systems