Row Echelon Form	Rank 00	Gaussian Elimination	References

Linear Methods (Math 211) - Lecture 2

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Row Echelon Form

Rank

Gaussian Elimination

References

Recall

Last time:

- Linear Systems
- Matrices
- Geometric Perspective
- Parametric Form

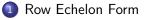
Row Echelon Form

Rank

Gaussian Elimination

References









Gaussian Elimination

Ran 00 References

Row Echelon Form

A matrix is in row echelon form if

- All zero rows are at the bottom;
- The first nonzero entry from the left in each nonzero row is a 1;
- Seach leading 1 is to the right of all leading 1s in the rows above it.
- A matrix is in reduced row echelon form if
 - Each leading 1 is the only nonzero entry in its column.

echelon form
$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix}$$
 $\begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$ reduced $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ reduced $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Row Echelon Form ○●○	Rank 00	Gaussian Elimination	

Solving Systems in Reduced Echelon Form

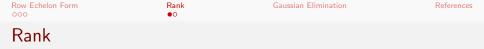
A linear system in reduced row echelon form is easy to solve:

$$\begin{bmatrix} 0 & 1 & 0 & 4 & | & 5 \\ 0 & 0 & 1 & 7 & | & 3 \\ 0 & 0 & 0 & 0 & | & 0 \end{bmatrix}$$
$$\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$
$$x_2 + 4x_4 = 5 \qquad x_1 = s$$
$$x_3 + 7x_4 = 3 \qquad x_2 = 5 - 4t$$
$$0 = 0 \qquad x_3 = 3 - 7t$$
$$x_4 = t$$

Row Echelon Form	Rank 00	Gaussian Elimination
Solving Systems in	Reduced Eche	elon Form

On the previous slide, x_2 and x_3 are leading variables, while x_1 and x_4 are nonleading variables. In general,

- If there is a row $\begin{bmatrix} 0 & 0 & \cdots & 0 & | & 1 \end{bmatrix}$ then the system is inconsistent, since this is the equation 0 = 1.
- Otherwise, each nonleading variable yields one parameter, and you can immediately solve for the leading variables in terms of the parameters.



Suppose A is any matrix. From A, different sequences of row operations can reach different row-echelon matrices. But there is precisely one *reduced* row-echelon matrix obtainable from A by a sequence of row operations. This matrix is known as *the* reduced row echelon form of A.

However, all of the row echelon forms of A share a feature in common: they all have the same number of leading 1s. This number is called the rank of A.

Note that if A has rank r, m rows and n columns. Then $r \le n$ and $r \le m$ (typo in book).

Row Echelon Form

Rank ⊙● Gaussian Elimination

References

Rank

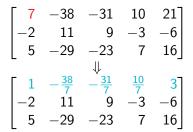
Theorem

Suppose a system of m equations in n variables is consistent, and that the rank of the augmented matrix is r.

- **1** The set of solutions involves exactly n r parameters,
- **2** If r < n, the system has infinitely many solutions,
- **()** If r = n, the system has a unique solution.

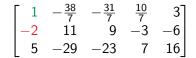
Row Echelon Form	Rank	Gaussian Elimination
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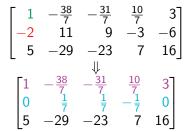
$$\begin{bmatrix} 7 & -38 & -31 & 10 & 21 \\ -2 & 11 & 9 & -3 & -6 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$



Multiply 1^{st} row by $\frac{1}{7}$

Row	Echelon	Form	
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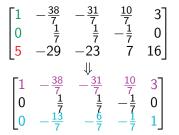


Add 2 times $1^{\rm st}$ row to $2^{\rm nd}$ row

Row	Echelon	Form
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References

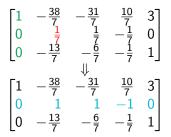
$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3\\ 0 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & 0\\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$



Add -5 times $1^{\rm st}$ row to $3^{\rm rd}$ row

Row	Echelon	Form
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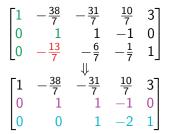




Multiply 2^{nd} row by 7

Row	Echelon	Form
000		

 $\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3\\ 0 & 1 & 1 & -1 & 0\\ 0 & -\frac{13}{7} & -\frac{6}{7} & -\frac{1}{7} & 1 \end{bmatrix}$



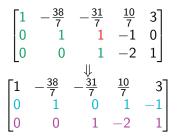
Add $\frac{13}{7}$ times $2^{\rm nd}$ row to $3^{\rm rd}$ row

Row	Echelon	Form	
000			

 $\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3\\ 0 & 1 & 1 & -1 & 0\\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$

Row	Echelon	Form
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References

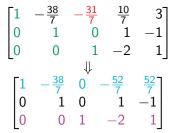


Add -1 times $3^{\rm rd}$ row to $2^{\rm nd}$ row

Row	Echelon	Form	
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 $\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3\\ 0 & 1 & 0 & 1 & -1\\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$

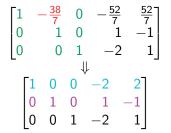
Row	Echelon	Form
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Add $\frac{31}{7}$ times $3^{\rm rd}$ row to $1^{\rm st}$ row

Row	Echelon	Form
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$$\begin{bmatrix} 1 & -\frac{38}{7} & 0 & -\frac{52}{7} & \frac{52}{7} \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$



Add $\frac{38}{7}$ times 2^{nd} row to 1^{st} row

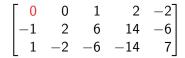
Computational Remarks

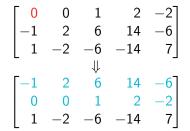
Rank

Row Echelon Form

- When solving a linear system on a computer, you get slight speedups by only reducing to a row echelon form and then substituting; this is known as back substitution
- Gaussian elimination is great for relatively small matrices, but there are faster algorithms for huge matrices, based on Strassen multiplication. [1, Algorithm 7.8; 2]
- When the numbers in the matrix are not known exactly, you need to be careful to reduce the growth of errors.
- The simplex method is based on Gaussian elimination and used for solving systems of linear inequalities. [3]

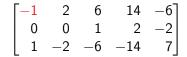
Row Echelon	Form
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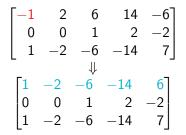




Swap $1^{\rm st}$ and $2^{\rm nd}$ rows

Row	Echelon	Form	
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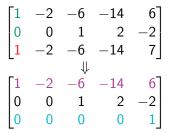




Multiply $1^{\rm st}$ row by -1

Row	Echelon	Form	
000			

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$

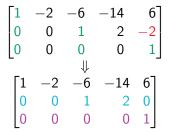


Add -1 times 1^{st} row to 3^{rd} row

Row	Echelon	Form
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$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Row	Echelon	Form
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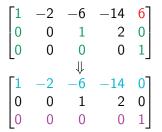


Add 2 times $3^{\rm rd}$ row to $2^{\rm nd}$ row

Row	Echelon	Form		
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Row	Echelon	Form
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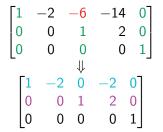


Add -6 times $3^{\rm rd}$ row to $1^{\rm st}$ row

Row	Echelon	Form
000		

 $\begin{bmatrix} 1 & -2 & -6 & -14 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

Row	Echelon	Form
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Add 6 times $2^{\rm nd}$ row to $1^{\rm st}$ row

Row Echelon Form

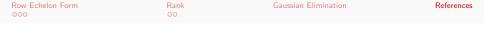
Rank

References

Practice

I've made an online applet that you can practice with. It's at http:

//people.ucalgary.ca/~roed/courses/211/practice.html,
or linked from the webpage.



- [1] William Stein. Linear Algebra Modular Forms, A Computational Approach. http://wstein.org/books/modform/modform/linear_algebra.html
- [2] Stoimen's Blog. Computer Algorithms: Strassen's Matrix Multiplication. http://www.stoimen.com/blog/2012/11/26/ computer-algorithms-strassens-matrix-multiplication/.
- [3] Simplex Algorithm. Wikipedia. http://en.wikipedia.org/wiki/Simplex_algorithm.