Linear Methods (Math 211) - Lecture 2

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Recall

Last time:

- Linear Systems
- Matrices
- Geometric Perspective
- Parametric Form

Today

Row Echelon Form

A matrix is in row echelon form if

1. All zero rows are at the bottom;
2. The first nonzero entry from the left in each nonzero row is a 1;
3. Each leading 1 is to the right of all leading 1s in the rows above it.

A matrix is in reduced row echelon form if

4. Each leading 1 is the only nonzero entry in its column.

echelon form
$$\begin{bmatrix} 1 & * & * \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & * & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{bmatrix}$$

reduced echelon form
$$\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Solving Systems in Reduced Echelon Form

A linear system in reduced row echelon form is easy to solve:

$$\left[\begin{array}{cccc|c} 0 & 1 & 0 & 4 & 5 \\ 0 & 0 & 1 & 7 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\begin{bmatrix} 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 0 \end{bmatrix}$$

$$\begin{aligned} x_2 + 4x_4 &= 5 \\ x_3 + 7x_4 &= 3 \\ 0 &= 0 \end{aligned}$$

$$\begin{aligned} x_1 &= s \\ x_2 &= 5 - 4t \\ x_3 &= 3 - 7t \\ x_4 &= t \end{aligned}$$

Solving Systems in Reduced Echelon Form

On the previous slide, x_2 and x_3 are leading variables, while x_1 and x_4 are nonleading variables. In general,

1. If there is a row $\left[\begin{array}{cccc|c} 0 & 0 & \cdots & 0 & 1 \end{array}\right]$ then the system is inconsistent, since this is the equation $0 = 1$.
2. Otherwise, each nonleading variable yields one parameter, and you can immediately solve for the leading variables in terms of the parameters.

Rank

Suppose A is any matrix. From A, different sequences of row operations can reach different row-echelon matrices. But there is precisely one *reduced* row-echelon matrix obtainable from A by a sequence of row operations. This matrix is known as *the* reduced row echelon form of A.

However, all of the row echelon forms of A share a feature in common: they all have the same number of leading 1s. This number is called the rank of A.

Note that if A has rank r, m rows and n columns. Then $r \leq n$ and $r \leq m$ (typo in book).

Rank

Theorem

Suppose a system of m equations in n variables is consistent, and that the rank of the augmented matrix is r.

1. *The set of solutions involves exactly $n - r$ parameters,*
2. *If $r < n$, the system has infinitely many solutions,*
3. *If $r = n$, the system has a unique solution.*

$$\begin{bmatrix} \color{red}{7} & -38 & -31 & 10 & 21 \\ -2 & 11 & 9 & -3 & -6 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 7 & -38 & -31 & 10 & 21 \\ -2 & 11 & 9 & -3 & -6 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ -2 & 11 & 9 & -3 & -6 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

Multiply 1^{st} row by $\frac{1}{7}$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ -2 & 11 & 9 & -3 & -6 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ -2 & 11 & 9 & -3 & -6 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & 0 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

Add 2 times 1^{st} row to 2^{nd} row

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & 0 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & 0 \\ 5 & -29 & -23 & 7 & 16 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & 0 \\ 0 & -\frac{13}{7} & -\frac{6}{7} & -\frac{1}{7} & 1 \end{bmatrix}$$

Add -5 times 1^{st} row to 3^{rd} row

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & 0 \\ 0 & -\frac{13}{7} & -\frac{6}{7} & -\frac{1}{7} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & \frac{1}{7} & \frac{1}{7} & -\frac{1}{7} & 0 \\ 0 & -\frac{13}{7} & -\frac{6}{7} & -\frac{1}{7} & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -\frac{13}{7} & -\frac{6}{7} & -\frac{1}{7} & 1 \end{bmatrix}$$

Multiply 2^{nd} row by 7

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -\frac{13}{7} & -\frac{6}{7} & -\frac{1}{7} & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & -\frac{13}{7} & -\frac{6}{7} & -\frac{1}{7} & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Add $\frac{13}{7}$ times 2^{nd} row to 3^{rd} row

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 1 & -1 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Add -1 times 3^{rd} row to 2^{nd} row

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & -\frac{31}{7} & \frac{10}{7} & 3 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & 0 & -\frac{52}{7} & \frac{52}{7} \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Add $\frac{31}{7}$ times 3^{rd} row to 1^{st} row

$$\begin{bmatrix} 1 & -\frac{38}{7} & 0 & -\frac{52}{7} & \frac{52}{7} \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -\frac{38}{7} & 0 & -\frac{52}{7} & \frac{52}{7} \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & 0 & 0 & -2 & 2 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}$$

Add $\frac{38}{7}$ times 2^{nd} row to 1^{st} row

Computational Remarks

- When solving a linear system on a computer, you get slight speedups by only reducing to a row echelon form and then substituting; this is known as back substitution
- Gaussian elimination is great for relatively small matrices, but there are faster algorithms for huge matrices, based on Strassen multiplication. [1, Algorithm 7.8; 2]
- When the numbers in the matrix are not known exactly, you need to be careful to reduce the growth of errors.
- The simplex method is based on Gaussian elimination and used for solving systems of linear inequalities. [3]

$$\begin{bmatrix} \color{red}{0} & 0 & 1 & 2 & -2 \\ -1 & 2 & 6 & 14 & -6 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 2 & -2 \\ -1 & 2 & 6 & 14 & -6 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} -1 & 2 & 6 & 14 & -6 \\ 0 & 0 & 1 & 2 & -2 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$

Swap 1^{st} and 2^{nd} rows

$$\begin{bmatrix} -1 & 2 & 6 & 14 & -6 \\ 0 & 0 & 1 & 2 & -2 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 2 & 6 & 14 & -6 \\ 0 & 0 & 1 & 2 & -2 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$

Multiply 1^{st} row by -1

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 1 & -2 & -6 & -14 & 7 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Add -1 times 1^{st} row to 3^{rd} row

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Add 2 times 3^{rd} row to 2^{nd} row

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 6 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$
$$\Downarrow$$
$$\begin{bmatrix} 1 & -2 & -6 & -14 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Add -6 times 3^{rd} row to 1^{st} row

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -6 & -14 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\Downarrow$$

$$\begin{bmatrix} 1 & -2 & 0 & -2 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Add 6 times 2^{nd} row to 1^{st} row

Practice

I've made an online applet that you can practice with. It's at `http://people.ucalgary.ca/~roed/courses/211/practice.html`, or linked from the webpage.

[1] William Stein. *Linear Algebra - Modular Forms, A Computational Approach*. http://wstein.org/books/modform/modform/linear_algebra.html

[2] Stoimen's Blog. *Computer Algorithms: Strassen's Matrix Multiplication*. http://www.stoimen.com/blog/2012/11/26/computer-algorithms-strassens-matrix-multiplication/.

[3] Simplex Algorithm. Wikipedia. http://en.wikipedia.org/wiki/Simplex_algorithm.