Linear Methods (Math 211)
Lecture 13 - §2.5 & 2.6

(with slides adapted from K. Seyffarth)

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Recall

1. Inverses of Elementary Matrices
2. Determining Elem. Matrices that Take A to B

Today

1. Products of Elementary Matrices
2. Linear Transformations
3. ... and Matrix Transformations

Example

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

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Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

First notice that A is invertible since $\det A = 8 - (-3) = 11 \neq 0$.

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \xrightarrow[\mathbf{e}_1]{} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \xrightarrow[\mathbf{e}_2]{} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \xrightarrow[\mathbf{e}_3]{} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[\mathbf{e}_4]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$\mathbf{e}_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}, \mathbf{e}_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

Since $E_4E_3E_2E_1A = I$, $A^{-1} = E_4E_3E_2E_1$, and hence

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}.$$

Example (continued)

Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1},$$

i.e.,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Problem

Is I_n an elementary matrix? Explain.

Problem

Is 0 an elementary matrix? Explain.

Theorem (§2.5 Theorem 4)

If A is a matrix, and R and S are reduced row-echelon forms of A, then $R = S$.

Linear Transformations

Definition

A transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ is a **linear transformation** if and only if, for all $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and all scalars a,

(T1) $T(\mathbf{x} + \mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ (preservation of addition)

(T2) $T(a\mathbf{x}) = aT(\mathbf{x})$ (preservation of scalar multiplication)

As a consequence of $T2$, for any linear transformation T,

$$T(0\mathbf{x}) = 0T(\mathbf{x}), \text{ implying } T(0) = 0,$$

and

$$T((-1)\mathbf{x}) = (-1)T(\mathbf{x}), \text{ implying } T(-\mathbf{x}) = -T(\mathbf{x}),$$

i.e., T preserves the zero vector and T preserves the negative of a vector.

Furthermore, if $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$ are vectors in $\mathbb{R}^n$ and $\mathbf{y}$ is a linear combination of $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_k$, i.e.,

$$\mathbf{y} = a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k$$

for some $a_1, a_2, \ldots, a_k \in \mathbb{R}$, then (T1) and (T2) used repeatedly give us

$$\begin{aligned} T(\mathbf{y}) &= T(a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_k\mathbf{x}_k) \\ &= a_1 T(\mathbf{x}_1) + a_2 T(\mathbf{x}_2) + \cdots + a_k T(\mathbf{x}_k), \end{aligned}$$

i.e., T preserves linear combinations.

Example

Let $T : \mathbb{R}^4 \to \mathbb{R}^3$ be a linear transformation such that

$$T\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \text{ and } T\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix}. \text{ Find } T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix}.$$

Example

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The only way it is possible to solve this problem is if

$$\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} \text{ is a linear combination of } \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix},$$

i.e., if there exist $a, b \in \mathbb{R}$ so that

$$\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} = a\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} + b\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}$$

Example (continued)

Solve the system of four equations in two variables:

$$\left[\begin{array}{rr|r} 1 & 0 & 1 \\ 1 & -1 & 3 \\ 0 & 1 & -2 \\ -2 & 1 & -4 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rr|r} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

Thus $a = 1$, $b = -2$, and

$$\left[\begin{array}{r} 1 \\ 3 \\ -2 \\ -4 \end{array}\right] = \left[\begin{array}{r} 1 \\ 1 \\ 0 \\ -2 \end{array}\right] - 2 \left[\begin{array}{r} 0 \\ -1 \\ 1 \\ 1 \end{array}\right].$$

Example (continued)

It follows that

$$
\begin{aligned}
T\begin{bmatrix} 1 \\ 3 \\ -2 \\ -4 \end{bmatrix} &= T\left(\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix}\right) \\
&= T\begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix} - 2T\begin{bmatrix} 0 \\ -1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -8 \\ 3 \\ -3 \end{bmatrix}
\end{aligned}
$$

Example (§2.6 Example 2)

Every matrix transformation is a linear transformation.

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Proof.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation induced by the $m \times n$ matrix A, i.e., $T(\mathbf{x}) = A\mathbf{x}$ for each $\mathbf{x} \in \mathbb{R}^n$.
Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ and let $a \in \mathbb{R}$. Then

$$T(\mathbf{x}+\mathbf{y}) = A(\mathbf{x}+\mathbf{y}) = A\mathbf{x} + A\mathbf{y} = T(\mathbf{x}) + T(\mathbf{y}),$$

proving that T preserves addition. Also,

$$T(a\mathbf{x}) = A(a\mathbf{x}) = a(A\mathbf{x}) = aT(\mathbf{x}),$$

proving that T preserves scalar multiplication.

Since (T1) and (T2) are satisfied, T is a linear transformation. □

It turns out that the converse of this statement is also true, i.e., every linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ is a matrix transformation.

Theorem (§2.6 Theorem 2)

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a transformation.

1. *T is linear if and only if T is a matrix transformation.*
2. *If T is linear, then T is induced by the unique matrix*

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix},$$

where $\mathbf{e}_j$ is the j^{th} column of I_n.

The uniqueness in Theorem 2 guarantees that there is exactly one matrix for any linear transformation, so it makes sense to say the matrix of a linear transformation.

Examples

Consider the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$.

- Let Q_0 be reflection across the x-axis.
- Let $R_{\pi/2}$ be rotation by $\frac{\pi}{2}$ counterclockwise.
- Let Q_1 be reflection across the line $y = x$.

Find the matrices associated to them using Theorem 2.

Examples

Consider the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$.

- Let Q_0 be reflection across the x-axis.
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- Let Q_1 be reflection across the line $y = x$.

Find the matrices associated to them using Theorem 2.

$$Q_0(\mathbf{e}_1) = Q_0\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$
$$Q_0(\mathbf{e}_2) = Q_0\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}0\\-1\end{bmatrix}$$

Thus the matrix for Q_0 is $\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}$.

Examples (continued)

$$R_{\pi/2}(\mathbf{e}_1) = R_{\pi/2}\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$R_{\pi/2}(\mathbf{e}_2) = R_{\pi/2}\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}-1\\0\end{bmatrix}$$

Thus the matrix for $R_{\pi/2}$ is $\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$.

$$Q_1(\mathbf{e}_1) = Q_1\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\1\end{bmatrix}$$

$$Q_1(\mathbf{e}_2) = Q_1\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}1\\0\end{bmatrix}$$

Thus the matrix for Q_1 is $\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$.

Summary

1. Products of Elementary Matrices
2. Linear Transformations
3. ... and Matrix Transformations