Linear Methods (Math 211)
Lecture 12 - §2.5

(with slides adapted from K. Seyffarth)

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Recall

1. Properties of Inversion
2. Inverses of Matrix Transformations
3. Elementary Matrices

Today

1. Inverses of Elementary Matrices
2. Determining Elem. Matrices that Take A to B

Inverses of Elementary Matrices

Example

Without using the matrix inversion algorithm, what is the inverse of the elementary matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}?$$

Inverses of Elementary Matrices

Example

Without using the matrix inversion algorithm, what is the inverse of the elementary matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}?$$

The row operation $G \rightarrow I_4$ is to add three times row one to row three,

and thus

$$G^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example (continued)

Similarly,

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} =$$

and

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} =$$

Example (continued)

Similarly,

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

and

$$F^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Suppose A is an $m \times n$ matrix and that B can be obtained from A by a sequence of k elementary row operations. Then there exist elementary matrices $E_1, E_2, \ldots E_k$ such that

$$\begin{aligned} B &= E_k(E_{k-1}(\cdots(E_2(E_1A))\cdots)) \\ &= (E_kE_{k-1}\cdots E_2E_1)A, \end{aligned}$$

Since matrix multiplication is associative, we have

$$B = (E_kE_{k-1}\cdots E_2E_1)A,$$

or $B = UA$ where $U = E_kE_{k-1}\cdots E_2E_1$.

To find U so that $B = UA$, we *could* find $E_1, E_2, \ldots, E_k$ and multiply these together (in the correct order), but there is an easier method for finding U.

Let A and B be $m \times n$ matrices. We write

$$A \rightarrow B$$

if B can be obtained from A by a sequence of elementary row operations.

Theorem (§2.5 Theorem 1)

Suppose A is an $m \times n$ matrix and that $A \rightarrow B$. Then

1. $B = UA$ *where* U *is an* $m \times m$ *matrix.*
2. U *can be computed by*

$$\left[A \mid I \right] \rightarrow \left[B \mid U \right].$$

3. $U = E_k E_{k-1} \cdots E_2 E_1$, *where* $E_1, E_2, \ldots, E_k$ *are elementary matrices corresponding, in order, to the elementary row operations used to transform* A *to* B.

Example

Let $A = \begin{bmatrix} 3 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix}$ and R be the reduced row-echelon form of A. Find a matrix U so that $R = UA$ and a matrix Q so that $A = QR$.

Example

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$$\left[\begin{array}{rrr|rr} 3 & 0 & 1 & 1 & 0 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 2 & -1 & 0 & 0 & 1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & -3 & -2 & -2 & 3 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{rrr|rr} 1 & 1 & 1 & 1 & -1 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -1 \end{array}\right] \rightarrow \left[\begin{array}{rrr|rr} 1 & 0 & \frac{1}{3} & \frac{1}{3} & 0 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} & -1 \end{array}\right]$$

Starting with $\left[\,A \mid I\,\right]$, we've obtained $\left[\,R \mid U\,\right]$.
Therefore $R = UA$, where

$$U = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{bmatrix}.$$

Since $R = UA$ and U is invertible, we have

Example (continued)

$$\begin{aligned} R &= UA \\ U^{-1}R &= U^{-1}(UA) \\ &= (U^{-1}U)A \\ &= IA \\ &= A. \end{aligned}$$

Thus $A = U^{-1}R$, and

$$Q = U^{-1} = \begin{bmatrix} \frac{1}{3} & 0 \\ \frac{2}{3} & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 3 & 0 \\ 2 & -1 \end{bmatrix}.$$

Example

Let $A = \begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix}$. Do row operations to put A in reduced row-echelon form, and write down the corresponding elementary matrices.

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$$\begin{bmatrix} 1 & 2 & -4 \\ -3 & -6 & 13 \\ 0 & -1 & 2 \end{bmatrix} \overrightarrow{E_1} \begin{bmatrix} 1 & 2 & -4 \\ 0 & 0 & 1 \\ 0 & -1 & 2 \end{bmatrix} \overrightarrow{E_2} \begin{bmatrix} 1 & 2 & -4 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \overrightarrow{E_3}$$

$$\begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \overrightarrow{E_4} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \overrightarrow{E_5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Example

$$E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$E_4 = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, E_5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

It follows that

$$\begin{aligned} (E_5(E_4(E_3(E_2(E_1A))))) &= I \\ (E_5E_4E_3E_2E_1)A &= I \end{aligned}$$

and therefore

$$A^{-1} = E_5E_4E_3E_2E_1.$$

Example (continued)

Since $A^{-1} = E_5E_4E_3E_2E_1$,

$$\begin{aligned} A^{-1} &= E_5E_4E_3E_2E_1 \\ (A^{-1})^{-1} &= (E_5E_4E_3E_2E_1)^{-1} \\ A &= E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}. \end{aligned}$$

This example illustrates the following result.

Theorem (§2.5 Theorem 2)

A square matrix is invertible if and only if it is the product of elementary matrices.

Example

Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

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Express $A = \begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix}$ as a product of elementary matrices.

First notice that A is invertible since $\det A = 8 - (-3) = 11 \neq 0$.

$$\begin{bmatrix} 4 & 1 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_1]{} \begin{bmatrix} 1 & 3 \\ -3 & 2 \end{bmatrix} \xrightarrow[E_2]{} \begin{bmatrix} 1 & 3 \\ 0 & 11 \end{bmatrix} \xrightarrow[E_3]{} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \xrightarrow[E_4]{} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$E_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, E_2 = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}, E_3 = \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}, E_4 = \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}.$$

Since $E_4 E_3 E_2 E_1 A = I$, $A^{-1} = E_4 E_3 E_2 E_1$, and hence

$$A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1}.$$

Example (continued)

Therefore,

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{11} \end{bmatrix}^{-1} \begin{bmatrix} 1 & -3 \\ 0 & 1 \end{bmatrix}^{-1},$$

i.e.,

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 11 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}.$$

Example

Let $A = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$. Find elementary matrices E and F so that $C = FEA$.

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Note. *The statement of the problem tells you that C can be obtained from A by a sequence of two elementary row operations.*

$$\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \underset{E}{\rightarrow} \begin{bmatrix} -1 & 1 \\ 1 & 2 \end{bmatrix} \underset{F}{\rightarrow} \begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix}$$

$$E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } F = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

Therefore $\begin{bmatrix} -1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$.

Problem

Is I_n an elementary matrix? Explain.

Problem

Is 0 an elementary matrix? Explain.

Theorem (§2.5 Theorem 4)

If A is a matrix, and R and S are reduced row-echelon forms of A, then $R = S$.

Summary

1. Inverses of Elementary Matrices
2. Determining Elem. Matrices that Take A to B