Linear Methods (Math 211)
Lecture 21 - §3.1

(with slides adapted from K. Seyffarth)

David Roe

October 30, 2013

Recall

1. Determinants
2. Elementary Row Operations

Today

Example

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -1$, find $\det \begin{bmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{bmatrix}$.

Example

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -1$, find $\det \begin{bmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{bmatrix}$.

$$\begin{vmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{vmatrix} = (-1)(2) \begin{vmatrix} x & y & z \\ 3p+a & 3q+b & 3r+c \\ p & q & r \end{vmatrix}$$

$$= -2 \begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = -2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$= (-2)(-1) = 2.$$

Example (Upper Triangular Matrix)

$$
\begin{aligned}
\det \begin{bmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 0 & 0 & 9 \end{bmatrix} &= 1 \det \begin{bmatrix} 5 & 6 \\ 0 & 9 \end{bmatrix} \\
&= (1)(5) \det \begin{bmatrix} 9 \end{bmatrix} \\
&= (1)(5)(9) \\
&= 45.
\end{aligned}
$$

Definitions

1. An $n \times n$ matrix A is called upper triangular if and only if all entries **below** the main diagonal are zero.
2. An $n \times n$ matrix A is called lower triangular if and only if all entries **above** the main diagonal are zero.
3. An $n \times n$ matrix A is called triangular if and only if it is upper triangular or lower triangular.

Theorem (§3.1 Theorem 4)

If $A = \left[a_{ij}\right]$ is an $n \times n$ triangular matrix, then

$$\det A = a_{11}a_{22}a_{33}\cdots a_{nn},$$

i.e., $\det A$ is the product of the entries of the main diagonal of A.

Example

Suppose A is a 3×3 matrix with $\det A = 7$. What is $\det(-3A)$?

Write $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. Then

$$-3A = \begin{bmatrix} -3a_{11} & -3a_{12} & -3a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{bmatrix}.$$

$$\begin{vmatrix} -3a_{11} & -3a_{12} & -3a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix} = (-3) \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ -3a_{21} & -3a_{22} & -3a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix}$$

$$= (-3)^2 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ -3a_{31} & -3a_{32} & -3a_{33} \end{vmatrix} = (-3)^3 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$= (-3)^3 \det A = (-27) \times 7 = -189.$$

Theorem (§3.1 Theorem 3)

If A is an $n \times n$ matrix and $k \in \mathbb{R}$ is a scalar, then

$$\det(kA) = k^n \det A.$$

Example

Let

$$A = \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} \text{ and } B = \begin{bmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{bmatrix}$$

Show that $\det B = 9 \det A$.

Example (continued)

$$\det B = \begin{vmatrix} 2a+p & 2b+q & 2c+r \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} = \begin{vmatrix} p-4x & q-4y & r-4z \\ 2p+x & 2q+y & 2r+z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$= \begin{vmatrix} p-4x & q-4y & r-4z \\ 9x & 9y & 9z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} = 9 \begin{vmatrix} p-4x & q-4y & r-4z \\ x & y & z \\ 2x+a & 2y+b & 2z+c \end{vmatrix}$$

$$= 9 \begin{vmatrix} p & q & r \\ x & y & z \\ 2x+a & 2y+b & 2z+c \end{vmatrix} = 9 \begin{vmatrix} p & q & r \\ x & y & z \\ a & b & c \end{vmatrix}$$

$$= -9 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = 9 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix} = 9 \det A.$$

Theorem (§3.1 Theorem 5)

Consider the matrices

$$\begin{bmatrix} A & X \\ 0 & B \end{bmatrix} \text{ and } \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix}$$

where A and B are square matrices. Then

$$\det \begin{bmatrix} A & X \\ 0 & B \end{bmatrix} = \det A \det B \text{ and } \det \begin{bmatrix} A & 0 \\ Y & B \end{bmatrix} = \det A \det B.$$

Example

$$\det \begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} =$$

Example

$$\det \begin{bmatrix} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} = \det \left[\begin{array}{ccc|cc} 1 & -1 & 2 & 0 & -2 \\ 0 & 1 & 0 & 4 & 1 \\ 1 & 1 & 5 & 0 & 0 \\ \hline 0 & 0 & 0 & 3 & -1 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

$$= \det \begin{bmatrix} 1 & -1 & 2 \\ 0 & 1 & 0 \\ 1 & 1 & 5 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 2 \\ 1 & 5 \end{bmatrix} \det \begin{bmatrix} 3 & -1 \\ 1 & -1 \end{bmatrix}$$

$$= 3 \times (-2) = -6.$$

Example (§3.1 Exercise 6(a))

Evaluate by inspection.

$$\det \begin{bmatrix} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{bmatrix} = \ ?$$

Example (§3.1 Exercise 6(a))

Evaluate by inspection.

$$\det \begin{bmatrix} a & b & c \\ a+1 & b+1 & c+1 \\ a-1 & b-1 & c-1 \end{bmatrix} = \ ?$$

$$\text{row2} + \text{row3} - 2(\text{row1}) = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

Example (§3.1 Exercise 13)

(a) Find $\det A$ if A is 3×3 and $\det(2A) = 6$.

(b) Let A be an $n \times n$ matrix. Under what conditions is $\det(-A) = \det A$?

Example (§3.1 Exercise 13)

(a) Find $\det A$ if A is 3×3 and $\det(2A) = 6$.

$$6 = \det(2A) = 2^3 \det A = 8 \det A$$

so $\det A = \frac{3}{4}$

(b) Let A be an $n \times n$ matrix. Under what conditions is $\det(-A) = \det A$?

$$\det(-A) = (-1)^n \det(A)$$

so they will be equal if and only if $(-1)^n = 1$, which occurs when n is even.

Example (§3.1 Exercise 9)

In each case, prove the statement is true or give a counterexample showing that the statement is false.

(a) $\det(A + B) = \det A + \det B$.

(c) If A is 2×2, then $\det(A^T) = \det A$.

(e) If A is 2×2, then $\det(7A) = 49 \det A$.

(g) $\det(-A) = -\det A$.

Example (§3.1 Exercise 9)

In each case, prove the statement is true or give a counterexample showing that the statement is false.

(a) $\det(A+B) = \det A + \det B$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

but $1 + 1 \neq 0$. False

(c) If A is 2×2, then $\det(A^T) = \det A$.
If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then both sides are $ad - bc$. True

(e) If A is 2×2, then $\det(7A) = 49 \det A$.
$\det(7A) = 7^2 \det A = 49 \det A$. True

(g) $\det(-A) = -\det A$.
$\det(-I_2) = (-1)^2 \det(I_2) = 1 \neq -\det(I_2)$. False

Summary

1. Elementary Row Operations
2. Triangular Matrices
3. Multiplying by scalars
4. Block Matrices
5. More examples