Linear Methods (Math 211)
Lecture 19 - Appendix A & 3.1

(with slides adapted from K. Seyffarth)

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Recall

1. Quadratic Polynomials
2. Complex Numbers: Final Notes
3. Determinants

Today

1. Determinants
2. Elementary Row Operations

Cofactor expansion

We define the determinant of an $n \times n$ matrix iteratively, in terms of determinants of $(n-1) \times (n-1)$ matrices. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- The sign of the (i, j) position is $(-1)^{i+j}$.
  Thus the sign is 1 if $(i + j)$ is even, and -1 if $(i + j)$ is odd.

Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.

- The (i, j)-cofactor of A is

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Finally,

- $\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + a_{13}c_{13}(A) + \cdots + a_{1n}c_{1n}(A)$, and is called the cofactor expansion of $\det A$ along row 1.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find $\det A$.

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Using cofactor expansion along row 1,

$$\begin{aligned}
\det A &= 1c_{11}(A) + 2c_{12}(A) + 3c_{13}(A) \\
&= 1(-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= (45 - 48) - 2(36 - 42) + 3(32 - 35) \\
&= -3 - 2(-6) + 3(-3) \\
&= -3 + 12 - 9 \\
&= 0
\end{aligned}$$

Example (continued)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now try cofactor expansion along column 2.

$$\begin{aligned} \det A &= 2c_{12}(A) + 5c_{22}(A) + 8c_{32}(A) \\ &= 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8(-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2(36 - 42) + 5(9 - 21) - 8(6 - 12) \\ &= -2(-6) + 5(-12) - 8(-6) \\ &= 12 - 60 + 48 \\ &= 0. \end{aligned}$$

We get the same answer!

Theorem (§3.1 Theorem 1)

The determinant of an $n \times n$ matrix A can be computed using the cofactor expansion along any row or column of A.

Example

Let $A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$. Find $\det A$.

Example

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Cofactor expansion along row 1 yields

$$\begin{aligned} \det A &= 0c_{11}(A) + 1c_{12}(A) + 2c_{13}(A) + 1c_{14}(A) \\ &= 1c_{12}(A) + 2c_{13}(A) + c_{14}(A), \end{aligned}$$

whereas cofactor expansion along, row 3 yields

$$\begin{aligned} \det A &= 0c_{31}(A) + 1c_{32}(A) + (-1)c_{33}(A) + 0c_{34}(A) \\ &= 1c_{32}(A) + (-1)c_{33}(A), \end{aligned}$$

i.e., in the first case we have to compute three cofactors, but in the second we only have to compute two.

Example (continued)

We use cofactor expansion along row 3 rather than row 1.

$$A = \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$$

$$\begin{aligned} \det A &= 1c_{32}(A) + (-1)c_{33}(A) \\ &= 1(-1)^5 \begin{vmatrix} 0 & 2 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} + (-1)(-1)^6 \begin{vmatrix} 0 & 1 & 1 \\ 5 & 0 & 7 \\ 3 & 0 & 2 \end{vmatrix} \\ &= (-1)2(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} + (-1)1(-1)^3 \begin{vmatrix} 5 & 7 \\ 3 & 2 \end{vmatrix} \\ &= 2(10-21) + 1(10-21) \\ &= 2(-11) + (-11) = -33. \end{aligned}$$

Example (continued)

Try computing $\det \begin{bmatrix} 0 & 1 & 2 & 1 \\ 5 & 0 & 0 & 7 \\ 0 & 1 & -1 & 0 \\ 3 & 0 & 0 & 2 \end{bmatrix}$ using cofactor expansion along other rows and columns, for instance column 2 or row 4. You will still get $\det A = -33$.

Example

Find $\det A$ for $A = \begin{bmatrix} -8 & 1 & 0 & -4 \\ 5 & 7 & 0 & -7 \\ 12 & -3 & 0 & 8 \\ -3 & 11 & 0 & 2 \end{bmatrix}$.

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Solution.
Using cofactor expansion along column 3, $\det A = 0$.

Example

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Solution.

Using cofactor expansion along column 3, $\det A = 0$.

Fact

If A is an $n \times n$ matrix with a row or column of zeros, then $\det A = 0$.

Elementary Row Operations and Determinants

Example

Let $A = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 1 & 0 & -2 \end{bmatrix}$. Then

$$\det A = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = 4(-1) = -4.$$

Let B_1, B_2, and B_3 be obtained from A by performing a type 1, 2 and 3 elementary row operation, respectively, i.e.,

$$B_1 = \begin{bmatrix} 2 & 0 & -3 \\ 1 & 0 & -2 \\ 0 & 4 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ -3 & 0 & 6 \end{bmatrix}, B_3 = \begin{bmatrix} 2 & 0 & -3 \\ 0 & 4 & 0 \\ 5 & 0 & -8 \end{bmatrix}.$$

Compute $\det B_1$, $\det B_2$, and $\det B_3$.

Elementary Row Operations and Determinants

Example (continued)

$$\det B_1 = 4(-1)^5 \begin{vmatrix} 2 & -3 \\ 1 & -2 \end{vmatrix} = (-4)(-1) = 4 = (-1)\det A.$$

$$\det B_2 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ -3 & 6 \end{vmatrix} = 4(12-9) = 4 \times 3 = 12 = -3\det A.$$

$$\det B_3 = 4(-1)^4 \begin{vmatrix} 2 & -3 \\ 5 & -8 \end{vmatrix} = 4(-16+15) = 4(-1) = -4 = \det A.$$

Theorem (§3.1 Theorem 2)

Let A be an $n \times n$ matrix.

1. *If A has a row or column of zeros, then* $\det A = 0$.
2. *If B is obtained from A by exchanging two different rows (or columns) of A, then* $\det B = -\det A$.
3. *If B is obtained from A by multiplying a row (or column) of A by a scalar $k \in \mathbb{R}$, then* $\det B = k \det A$.
4. *If B is obtained from A by adding k times one row of A to a different row of A (or adding k times one column of A to a different column of A) then* $\det B = \det A$.
5. *If two different rows (or columns) of A are identical, then* $\det A = 0$.

Example

$$\det \begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} =$$

Example

$$\det\begin{bmatrix} 3 & 1 & 2 & 4 \\ -1 & -3 & 8 & 0 \\ 1 & -1 & 5 & 5 \\ 1 & 1 & 2 & -1 \end{bmatrix} = \begin{vmatrix} 0 & -8 & 26 & 4 \\ -1 & -3 & 8 & 0 \\ 0 & -4 & 13 & 5 \\ 0 & -2 & 10 & -1 \end{vmatrix}$$

$$= (-1)(-1)^3 \begin{vmatrix} -8 & 26 & 4 \\ -4 & 13 & 5 \\ -2 & 10 & -1 \end{vmatrix}$$

$$= \begin{vmatrix} 0 & -14 & 8 \\ 0 & -7 & 7 \\ -2 & 10 & -1 \end{vmatrix}$$

$$= (-2)(-1)^4 \begin{vmatrix} -14 & 8 \\ -7 & 7 \end{vmatrix}$$

$$= -2 \begin{vmatrix} 0 & -6 \\ -7 & 7 \end{vmatrix}$$

$$= (-2)(-42) = 84.$$

Example

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -1$, find $\det \begin{bmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{bmatrix}$.

Example

If $\det \begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix} = -1$, find $\det \begin{bmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{bmatrix}$.

$$\begin{vmatrix} -x & -y & -z \\ 3p+a & 3q+b & 3r+c \\ 2p & 2q & 2r \end{vmatrix} = (-1)(2) \begin{vmatrix} x & y & z \\ 3p+a & 3q+b & 3r+c \\ p & q & r \end{vmatrix}$$

$$= -2 \begin{vmatrix} x & y & z \\ a & b & c \\ p & q & r \end{vmatrix} = 2 \begin{vmatrix} a & b & c \\ x & y & z \\ p & q & r \end{vmatrix} = -2 \begin{vmatrix} a & b & c \\ p & q & r \\ x & y & z \end{vmatrix}$$

$$= (-2)(-1) = 2.$$

Summary

1. Determinants
2. Elementary Row Operations