Linear Methods (Math 211)
Lecture 19 - Appendix A & 3.1

(with slides adapted from K. Seyffarth)

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Recall

1. Roots of Unity
2. Finding Roots
3. Quadratic Polynomials
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Example

The quadratic $x^2 - 14x + 58$ has roots

$$x = \frac{14 \pm \sqrt{196 - 4 \times 58}}{2}$$

$$= \frac{14 \pm \sqrt{196 - 232}}{2}$$

$$= \frac{14 \pm \sqrt{-36}}{2}$$

$$= \frac{14 \pm 6i}{2}$$

$$= 7 \pm 3i,$$

so the roots are $7 + 3i$ and $7 - 3i$. 
Conversely, given $u = a + bi$ with $b \neq 0$, there is an irreducible quadratic having roots $u$ and $\bar{u}$.

**Example**

Find an irreducible quadratic with $u = 5 - 2i$ as a root. What is the other root?
Conversely, given $u = a + bi$ with $b \neq 0$, there is an irreducible quadratic having roots $u$ and $\bar{u}$.

**Example**

Find an irreducible quadratic with $u = 5 - 2i$ as a root. What is the other root?

**Solution.**

\[
(x - u)(x - \bar{u}) = (x - (5 - 2i))(x - (5 + 2i))
= x^2 - (5 - 2i)x - (5 + 2i)x + (5 - 2i)(5 + 2i)
= x^2 - 10x + 29.
\]

Therefore, $x^2 - 10x + 29$ is an irreducible quadratic with roots $5 - 2i$ and $5 + 2i$.

Notice that $-10 = -(u + \bar{u})$ and $29 = u\bar{u} = |u|^2$. 
Example

Find the roots of the quadratic \( x^2 - (3 - 2i)x + (5 - i) = 0 \).
Quadratics with Complex Coefficients

Example

Find the roots of the quadratic $x^2 - (3 - 2i)x + (5 - i) = 0$.

Using the quadratic formula

$$x = \frac{3 - 2i \pm \sqrt{(-(3 - 2i))^2 - 4(5 - i)}}{2}$$

Now,

$$(-(3 - 2i))^2 - 4(5 - i) = 5 - 12i - 20 + 4i = -15 - 8i,$$

so

$$x = \frac{3 - 2i \pm \sqrt{-15 - 8i}}{2}$$

To find $\pm \sqrt{-15 - 8i}$, solve $z^2 = -15 - 8i$ for $z$. 
Example (continued)

Let \( z = a + bi \) and \( z^2 = -15 - 8i \). Then

\[
(a^2 - b^2) + 2abi = -15 - 8i,
\]

so \( a^2 - b^2 = -15 \) and \( 2ab = -8 \).

Solving for \( a \) and \( b \) gives us \( z = \pm (1 - 4i) \).

Therefore,

\[
x = \frac{3 - 2i \pm (1 - 4i)}{2};
\]

\[
\frac{3 - 2i + (1 - 4i)}{2} = \frac{4 - 6i}{2} = 2 - 3i,
\]

\[
\frac{3 - 2i - (1 - 4i)}{2} = \frac{2 + 2i}{2} = 1 + i.
\]

Thus the roots of \( x^2 - (3 - 2i)x + (5 - i) \) are \( 2 - 3i \) and \( 1 + i \).
Example

Verify that \( u_1 = (4 - i) \) is a root of

\[ x^2 - (2 - 3i)x - (10 + 6i) \]

and find the other root, \( u_2 \).
Example

Verify that $u_1 = (4 - i)$ is a root of

$$x^2 - (2 - 3i)x - (10 + 6i)$$

and find the other root, $u_2$.

$$u_1^2 - (2 - 3i)u_1 - (10 + 6i)$$

$$= (4 - i)^2 - (2 - 3i)(4 - i) - (10 + 6i)$$

$$= (15 - 8i) - (5 - 14i) - (10 + 6i)$$

$$= 0,$$

so $u_1 = (4 - i)$ is a root.
Example (continued)

Recall that if $u_1$ and $u_2$ are the roots of the quadratic, then

$$u_1 + u_2 = (2 - 3i) \text{ and } u_1 u_2 = -(10 + 6i).$$

Since $u_1 = 4 - i$ and $u_1 + u_2 = 2 - 3i$,

$$u_2 = 2 - 3i - u_1 = 2 - 3i - (4 - i) = -2 - 2i.$$

Therefore, the other root is $u_2 = -2 - 2i$.

You can easily verify your answer by computing $u_1 u_2$:

$$u_1 u_2 = (4 - i)(-2 - 2i) = -10 - 6i = -(10 + 6i).$$
Fundamental Theorem of Algebra

Theorem

Suppose that $P(z)$ is a polynomial with complex coefficients. Then $P$ factors as a product of linear polynomials.

As a consequence, every polynomial of degree $n$ has $n$ roots (up to multiplicity).

Example

- $z^4 - 1 = (z - 1)(z + 1)(z - i)(z + i)$
- $z^2 - 2z + 1 = (z - 1)^2$
Complex Exponentials

The equation \( e^{i\theta} = \cos \theta + i \sin \theta \) is more than just notation. One defining feature of exponentials is that \( e^{A+B} = e^A e^B \). So does \( e^{i(\theta+\phi)} = e^{i\theta} e^{i\phi} \)?

\[
(\cos \theta + i \sin \theta)(\cos \phi + i \sin \phi) = (\cos \theta \cos \phi - \sin \theta \sin \phi) + i(\cos \theta \sin \phi + \sin \theta \cos \phi) = \cos(\theta + \phi) + i \sin(\theta + \phi).
\]

To prove that \( e^{i\theta} = \cos \theta + i \sin \theta \), one can show that both sides satisfy the same differential equation with the same initial conditions

1. \( F'(\theta) = iF(\theta) \),
2. \( F(0) = 1 \).
Key Features of Determinants

Recall the determinant \( \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc \) of a \( 2 \times 2 \) matrix. In Chapter 3 we study the generalization to \( n \times n \) matrices.

- \( A \) is invertible if and only if \( \det(A) \neq 0 \).
- Thus \( \det(A) = 0 \) when there is a nontrivial solution to \( Ax = 0 \).
- \( \det(AB) = \det(A) \cdot \det(B) \).
- \(|\det(A)|\) is the amount by which the matrix transformation induced by \( A \) expands volumes.
- \( \det(A) \) is positive if and only if \( A \) preserves orientation.

We will see other properties and applications of determinants over the course of the chapter.
Cofactor expansion

We define the determinant of an $n \times n$ matrix iteratively, in terms of determinants of $(n-1) \times (n-1)$ matrices. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- The sign of the $(i, j)$ position is $(-1)^{i+j}$.

Thus the sign is 1 if $(i + j)$ is even, and $-1$ if $(i + j)$ is odd.

Let $A_{ij}$ denote the $(n-1) \times (n-1)$ matrix obtained from $A$ by deleting row $i$ and column $j$.

- The $(i, j)$-cofactor of $A$ is

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Finally,

- $\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + a_{13}c_{13}(A) + \cdots + a_{1n}c_{1n}(A)$, and is called the cofactor expansion of $\det A$ along row 1.
### Example

Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \). Find \( \det A \).
Example

Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \). Find det \( A \).

Using cofactor expansion along row 1,

\[
\text{det } A = 1c_{11}(A) + 2c_{12}(A) + 3c_{13}(A)
\]

\[
= 1(-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}
\]

\[
= (45 - 48) - 2(36 - 42) + 3(32 - 35)
\]

\[
= -3 - 2(-6) + 3(-3)
\]

\[
= -3 + 12 - 9
\]

\[
= 0
\]
Example

Let \( A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \). Find \( \det A \).

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= -3 - 2(-6) + 3(-3)
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\[
= -3 + 12 - 9
\]

\[
= 0
\]
Example (continued)

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \]

Now try cofactor expansion along column 2.

\[
\det A = 2c_{12}(A) + 5c_{22}(A) + 8c_{32}(A)
\]

\[
= 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8(-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix}
\]

\[
= -2(36 - 42) + 5(9 - 21) - 8(6 - 12)
\]

\[
= -2(-6) + 5(-12) - 8(-6)
\]

\[
= 12 - 60 + 48
\]

\[
= 0.
\]

We get the same answer!
Theorem (§3.1 Theorem 1)

The determinant of an $n \times n$ matrix $A$ can be computed using the cofactor expansion along any row or column of $A$. 
Summary

1. Quadratic Polynomials
2. Complex Numbers: Final Notes
3. Determinants