Linear Methods (Math 211)
Lecture 19 - Appendix A & 3.1

(with slides adapted from K. Seyffarth)

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October 23, 2013

Recall

1. Roots of Unity
2. Finding Roots
3. Quadratic Polynomials

Today

1. Quadratic Polynomials
2. Complex Numbers: Final Notes
3. Determinants

Example

The quadratic $x^2 - 14x + 58$ has roots

$$\begin{aligned} x &= \frac{14 \pm \sqrt{196 - 4 \times 58}}{2} \\ &= \frac{14 \pm \sqrt{196 - 232}}{2} \\ &= \frac{14 \pm \sqrt{-36}}{2} \\ &= \frac{14 \pm 6i}{2} \\ &= 7 \pm 3i, \end{aligned}$$

so the roots are $7 + 3i$ and $7 - 3i$.

Conversely, given $u = a + bi$ with $b \neq 0$, there is an irreducible quadratic having roots u and $\overline{u}$.

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Find an irreducible quadratic with $u = 5 - 2i$ as a root. What is the other root?

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Solution.

$$\begin{aligned}(x-u)(x-\overline{u}) &= (x-(5-2i))(x-(5+2i)) \\ &= x^2 - (5-2i)x - (5+2i)x + (5-2i)(5+2i) \\ &= x^2 - 10x + 29.\end{aligned}$$

Therefore, $x^2 - 10x + 29$ is an irreducible quadratic with roots $5 - 2i$ and $5 + 2i$.
Notice that $-10 = -(u + \overline{u})$ and $29 = u\overline{u} = |u|^2$.

Quadratics with Complex Coefficients

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Using the quadratic formula

$$x = \frac{3 - 2i \pm \sqrt{(-(3-2i))^2 - 4(5-i)}}{2}$$

Now,

$$(-(3-2i))^2 - 4(5-i) = 5 - 12i - 20 + 4i = -15 - 8i,$$

so

$$x = \frac{3 - 2i \pm \sqrt{-15-8i}}{2}$$

To find $\pm\sqrt{-15-8i}$, solve $z^2 = -15 - 8i$ for z.

Example (continued)

Let $z = a + bi$ and $z^2 = -15 - 8i$. Then

$$(a^2 - b^2) + 2abi = -15 - 8i,$$

so $a^2 - b^2 = -15$ and $2ab = -8$.

Solving for a and b gives us $z = \pm(1 - 4i)$.

Therefore,

$$x = \frac{3 - 2i \pm (1 - 4i)}{2};$$

$$\frac{3 - 2i + (1 - 4i)}{2} = \frac{4 - 6i}{2} = 2 - 3i,$$
$$\frac{3 - 2i - (1 - 4i)}{2} = \frac{2 + 2i}{2} = 1 + i.$$

Thus the roots of $x^2 - (3 - 2i)x + (5 - i)$ are $2 - 3i$ and $1 + i$.

Example

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$$\begin{aligned} u_1^2 - (2 - 3i)u_1 - (10 + 6i) & \\ &= (4 - i)^2 - (2 - 3i)(4 - i) - (10 + 6i) \\ &= (15 - 8i) - (5 - 14i) - (10 + 6i) \\ &= 0, \end{aligned}$$

so $u_1 = (4 - i)$ is a root.

Example (continued)

Recall that if u_1 and u_2 are the roots of the quadratic, then

$$u_1 + u_2 = (2 - 3i) \text{ and } u_1 u_2 = -(10 + 6i).$$

Since $u_1 = 4 - i$ and $u_1 + u_2 = 2 - 3i$,

$$u_2 = 2 - 3i - u_1 = 2 - 3i - (4 - i) = -2 - 2i.$$

Therefore, the other root is $u_2 = -2 - 2i$.

You can easily verify your answer by computing $u_1 u_2$:

$$u_1 u_2 = (4 - i)(-2 - 2i) = -10 - 6i = -(10 + 6i).$$

Fundamental Theorem of Algebra

Theorem

Suppose that $P(z)$ is a polynomial with complex coefficients. Then P factors as a product of linear polynomials.

As a consequence, every polynomial of degree n has n roots (up to multiplicity).

Example

- $z^4 - 1 = (z-1)(z+1)(z-i)(z+i)$
- $z^2 - 2z + 1 = (z-1)^2$

Complex Exponentials

The equation $e^{i\theta} = \cos\theta + i\sin\theta$ is more than just notation.
One defining feature of exponentials is that $e^{A+B} = e^A e^B$. So does

$$e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}?$$

$$\begin{aligned}(\cos\theta + i\sin\theta)&(\cos\phi + i\sin\phi)\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\cos\theta\sin\phi + \sin\theta\cos\phi)\\ &= \cos(\theta+\phi) + i\sin(\theta+\phi).\end{aligned}$$

To *prove* that $e^{i\theta} = \cos\theta + i\sin\theta$, one can show that both sides satisfy the same differential equation with the same initial conditions

- $F'(\theta) = iF(\theta)$,
- $F(0) = 1$.

Key Features of Determinants

Recall the determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$ of a 2×2 matrix. In Chapter 3 we study the generalization to $n \times n$ matrices.

- A is invertible if and only if $\det(A) \neq 0$.
- Thus $\det(A) = 0$ when there is a nontrivial solution to $A\mathbf{x} = \mathbf{0}$.
- $\det(AB) = \det(A) \cdot \det(B)$.
- $|\det(A)|$ is the amount by which the matrix transformation induced by A expands volumes.
- $\det(A)$ is positive if and only if A preserves orientation.

We will see other properties and applications of determinants over the course of the chapter.

Cofactor expansion

We define the determinant of an $n \times n$ matrix iteratively, in terms of determinants of $(n-1) \times (n-1)$ matrices. Let $A = [a_{ij}]$ be an $n \times n$ matrix.

- The sign of the (i, j) position is $(-1)^{i+j}$.
  Thus the sign is 1 if $(i + j)$ is even, and -1 if $(i + j)$ is odd.

Let A_{ij} denote the $(n-1) \times (n-1)$ matrix obtained from A by deleting row i and column j.

- The (i, j)-cofactor of A is

$$c_{ij}(A) = (-1)^{i+j} \det(A_{ij}).$$

Finally,

- $\det A = a_{11}c_{11}(A) + a_{12}c_{12}(A) + a_{13}c_{13}(A) + \cdots + a_{1n}c_{1n}(A)$, and is called the cofactor expansion of $\det A$ along row 1.

Example

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$. Find $\det A$.

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Using cofactor expansion along row 1,

$$\begin{aligned}
\det A &= 1c_{11}(A) + 2c_{12}(A) + 3c_{13}(A) \\
&= 1(-1)^2 \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} + 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3(-1)^4 \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\
&= (45 - 48) - 2(36 - 42) + 3(32 - 35) \\
&= -3 - 2(-6) + 3(-3) \\
&= -3 + 12 - 9 \\
&= 0
\end{aligned}$$

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Example (continued)

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now try cofactor expansion along column 2.

$$\begin{aligned} \det A &= 2c_{12}(A) + 5c_{22}(A) + 8c_{32}(A) \\ &= 2(-1)^3 \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 5(-1)^4 \begin{vmatrix} 1 & 3 \\ 7 & 9 \end{vmatrix} + 8(-1)^5 \begin{vmatrix} 1 & 3 \\ 4 & 6 \end{vmatrix} \\ &= -2(36 - 42) + 5(9 - 21) - 8(6 - 12) \\ &= -2(-6) + 5(-12) - 8(-6) \\ &= 12 - 60 + 48 \\ &= 0. \end{aligned}$$

We get the same answer!

Theorem (§3.1 Theorem 1)

The determinant of an $n \times n$ matrix A can be computed using the cofactor expansion along any row or column of A.

Summary

1. Quadratic Polynomials
2. Complex Numbers: Final Notes
3. Determinants