Linear Methods (Math 211)
Lecture 18 - Appendix A

(with slides adapted from K. Seyffarth)

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Recall

1. Properties of Absolute Value
2. The Complex Plane
3. Polar Form

Today

1. Roots of Unity
2. Finding Roots
3. Quadratic Polynomials

Roots of Unity

Example

Find **all** complex number z so that $z^3 = 1$, i.e., find the cube roots of unity. Express each root in the form $a + bi$.

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Let $z = re^{i\theta}$. Since $1 = 1e^{i0}$ in polar form, we want to solve

$$\left(re^{i\theta}\right)^3 = 1e^{i0},$$

i.e.,

$$r^3 e^{i3\theta} = 1e^{i0}.$$

Thus $r^3 = 1$ and $3\theta = 0 + 2\pi k = 2\pi k$ for some integer k.
Since $r^3 = 1$ and r is real, $r = 1$.

Example (continued)

Now $3\theta = 2\pi k$, so $\theta = \frac{2\pi}{3}k$.

k	θ	$e^{i\theta}$	
-3	-2π	$e^{-2\pi i}$	$= 1$
-2	$-\frac{4}{3}\pi$	$e^{(-4\pi/3)i}$	$= e^{(2\pi/3)i}$
-1	$-\frac{2}{3}\pi$	$e^{(-2\pi/3)i}$	$= e^{(-2\pi/3)i}$
0	0	e^{0i}	$= 1$
1	$\frac{2}{3}\pi$	$e^{(2\pi/3)i}$	$= e^{(2\pi/3)i}$
2	$\frac{4}{3}\pi$	$e^{(4\pi/3)i}$	$= e^{(-2\pi/3)i}$
3	2π	$e^{2\pi i}$	$= 1$

The three cube roots of unity are

$$\begin{aligned} e^{0\pi i} & & &= 1 \\ e^{(2\pi/3)i} &= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} & &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ e^{(-2\pi/3)i} &= \cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3} & &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

Theorem (Appendix A, Theorem 3 – n^{th} Roots of Unity)

For $n \geq 1$, the (complex) solutions to $z^n = 1$ are

$$z = e^{(2\pi k/n)i} \text{ for } k = 0, 1, 2, \ldots, n-1.$$

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For example, the sixth roots of unity are

$$z = e^{(2\pi k/6)i} = e^{(\pi k/3)i} \text{ for } k = 0, 1, 2, 3, 4, 5.$$

k	z	
0	e^{0i}	$= 1$
1	$e^{(\pi/3)i}$	$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$
2	$e^{(2\pi/3)i}$	$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
3	$e^{\pi i}$	$= -1$
4	$e^{(4\pi/3)i}$	$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
5	$e^{(5\pi/3)i}$	$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$

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First, convert $2(\sqrt{3}i - 1) = -2 + 2\sqrt{3}i$ to polar form:

$$|z^4| = \sqrt{(-2)^2 + (2\sqrt{3})^2} = \sqrt{16} = 4.$$

If $\phi = \arg(z^4)$, then

$$\cos\phi = \frac{-2}{4} = \frac{-1}{2} \qquad \sin\phi = \frac{2\sqrt{3}}{4} = \frac{\sqrt{3}}{2}$$

Thus, $\phi = \frac{2\pi}{3}$, and

$$z^4 = 4e^{(2\pi/3)i}.$$

Example (continued)

So $z^4 = 4e^{(2\pi/3)i}$.

Let $z = re^{i\theta}$. Then $z^4 = r^4 e^{i4\theta}$, so $r^4 = 4$ and $4\theta = \frac{2}{3}\pi + 2\pi k$ for $k = 0, 1, 2,$ or 3.

Since $r^4 = 4$, $r^2 = \pm 2$. But r is real, and so $r^2 = 2$, implying $r = \pm\sqrt{2}$. However $r \geq 0$, and therefore $r = \sqrt{2}$.

Since $4\theta = \frac{2}{3}\pi + 2\pi k$, $k = 0, 1, 2, 3$,

$$\begin{aligned}\theta &= \frac{2\pi}{12} + \frac{2\pi k}{4}\\ &= \frac{\pi}{6} + \frac{\pi k}{2}\\ &= \frac{\pi(3k+1)}{6}\end{aligned}$$

for $k = 0, 1, 2, 3$.

Example (continued)

$r = \sqrt{2}$ and $\theta = \left(\frac{3k+1}{6}\right)\pi,\ k = 0, 1, 2, 3.$

$$\begin{array}{llll}
k=0: & z=\sqrt{2}e^{(\pi/6)i} & =\sqrt{2}\left(\ \frac{\sqrt{3}}{2}+\frac{1}{2}i\right) & = \frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i \\
k=1: & z=\sqrt{2}e^{(2\pi/3)i} & =\sqrt{2}\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right) & = -\frac{\sqrt{2}}{2}+\frac{\sqrt{6}}{2}i \\
k=2: & z=\sqrt{2}e^{(7\pi/6)i} & =\sqrt{2}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) & = -\frac{\sqrt{6}}{2}-\frac{\sqrt{2}}{2}i \\
k=3: & z=\sqrt{2}e^{(5\pi/3)i} & =\sqrt{2}\left(\ \frac{1}{2}-\frac{\sqrt{3}}{2}i\right) & = \frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i
\end{array}$$

Therefore, the fourth roots of $2(\sqrt{3}i - 1)$ are:

$$\pm\left(\frac{\sqrt{6}}{2}+\frac{\sqrt{2}}{2}i\right) \text{ and } \pm\left(\frac{\sqrt{2}}{2}-\frac{\sqrt{6}}{2}i\right).$$

Real Quadratics

Definition

A quadratic is an expression of the form $ax^2 + bx + c$ where $a \neq 0$.

To find the roots of a quadratic, we can use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The expression $b^2 - 4ac$ in the quadratic formula is called the discriminant. If $a, b, c \in \mathbb{R}$ then we call $ax^2 + bx + c$ a real quadratic. In this case,

- if $b^2 - 4ac \geq 0$, then the roots of the quadratic are real;
- if $b^2 - 4ac < 0$, then the roots of the quadratic are complex conjugates of each other. In this case we call the quadratic irreducible.

Example

The quadratic $x^2 - 14x + 58$ has roots

$$\begin{aligned} x &= \frac{14 \pm \sqrt{196 - 4 \times 58}}{2} \\ &= \frac{14 \pm \sqrt{196 - 232}}{2} \\ &= \frac{14 \pm \sqrt{-36}}{2} \\ &= \frac{14 \pm 6i}{2} \\ &= 7 \pm 3i, \end{aligned}$$

so the roots are $7 + 3i$ and $7 - 3i$.

Conversely, given $u = a + bi$ with $b \neq 0$, there is an irreducible quadratic having roots u and $\overline{u}$.

Example

Find an irreducible quadratic with $u = 5 - 2i$ as a root. What is the other root?

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Example

Find an irreducible quadratic with $u = 5 - 2i$ as a root. What is the other root?

Solution.

$$\begin{aligned}(x-u)(x-\overline{u}) &= (x-(5-2i))(x-(5+2i)) \\ &= x^2 - (5-2i)x - (5+2i)x + (5-2i)(5+2i) \\ &= x^2 - 10x + 29.\end{aligned}$$

Therefore, $x^2 - 10x + 29$ is an irreducible quadratic with roots $5 - 2i$ and $5 + 2i$.
Notice that $-10 = -(u + \overline{u})$ and $29 = u\overline{u} = |u|^2$.

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Solution.

$$(x - u)(x - \overline{u}) = (x - (-3 + 4i))(x - (-3 - 4i))$$
$$= x^2 + 6x + 25.$$

Thus $x^2 + 6x + 25$ has roots $-3 + 4i$ and $-3 - 4i$.

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Using the quadratic formula

$$x = \frac{3-2i \pm \sqrt{(-(3-2i))^2 - 4(5-i)}}{2}$$

Now,

$$(-(3-2i))^2 - 4(5-i) = 5 - 12i - 20 + 4i = -15 - 8i,$$

so

$$x = \frac{3-2i \pm \sqrt{-15-8i}}{2}$$

To find $\pm\sqrt{-15-8i}$, solve $z^2 = -15 - 8i$ for z.

Example (continued)

Let $z = a + bi$ and $z^2 = -15 - 8i$. Then

$$(a^2 - b^2) + 2abi = -15 - 8i,$$

so $a^2 - b^2 = -15$ and $2ab = -8$.

Solving for a and b gives us $z = \pm(1 - 4i)$.

Therefore,

$$x = \frac{3 - 2i \pm (1 - 4i)}{2};$$

$$\frac{3 - 2i + (1 - 4i)}{2} = \frac{4 - 6i}{2} = 2 - 3i,$$
$$\frac{3 - 2i - (1 - 4i)}{2} = \frac{2 + 2i}{2} = 1 + i.$$

Thus the roots of $x^2 - (3 - 2i)x + (5 - i)$ are $2 - 3i$ and $1 + i$.

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$$x^2 - (2 - 3i)x - (10 + 6i)$$

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$$\begin{aligned} u_1^2 - (2 - 3i)u_1 - (10 + 6i) \\ &= (4 - i)^2 - (2 - 3i)(4 - i) - (10 + 6i) \\ &= (15 - 8i) - (5 - 14i) - (10 + 6i) \\ &= 0, \end{aligned}$$

so $u_1 = (4 - i)$ is a root.

Example (continued)

Recall that if u_1 and u_2 are the roots of the quadratic, then

$$u_1 + u_2 = (2 - 3i) \text{ and } u_1 u_2 = -(10 + 6i).$$

Since $u_1 = 4 - i$ and $u_1 + u_2 = 2 - 3i$,

$$u_2 = 2 - 3i - u_1 = 2 - 3i - (4 - i) = -2 - 2i.$$

Therefore, the other root is $u_2 = -2 - 2i$.

You can easily verify your answer by computing $u_1 u_2$:

$$u_1 u_2 = (4 - i)(-2 - 2i) = -10 - 6i = -(10 + 6i).$$

Summary

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3. Quadratic Polynomials