Linear Methods (Math 211)
Lecture 17 - Appendix A

(with slides adapted from K. Seyffarth)

David Roe

October 21, 2013

Recall

1. Complex Numbers: Basic Definitions
2. Arithmetic with Complex Numbers
3. Conjugates and Division

Today

1. Properties of Absolute Value
2. The Complex Plane
3. Polar Form
4. Roots of Unity

Recall

Suppose that $z = a + bi$ is a complex number.

- The conjugate of z is the complex number

$$\overline{z} = a - bi.$$

- The absolute value or modulus of z is

$$|z| = \sqrt{a^2 + b^2}.$$

Properties of the Conjugate and Absolute Value (p. 507)

Let z and w be complex numbers.

C1. $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

C2. $\overline{(zw)} = \overline{z}\ \overline{w}$.

C3. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.

C4. $\overline{(\overline{z})} = z$.

C5. z is real if and only if $\overline{z} = z$.

C6. $z \cdot \overline{z} = |z|^2$.

C7. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

C8. $|z| \geq 0$ for all complex numbers z

C9. $|z| = 0$ if and only if $z = 0$.

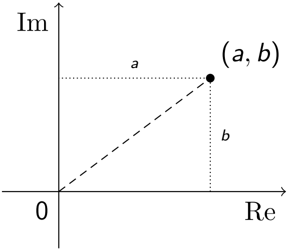
C10. $|zw| = |z|\ |w|$.

C11. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

C12. **Triangle Inequality** $|z + w| \leq |z| + |w|$.

The Complex Plane

Represent $z = a + bi$ as a point (a, b) in the plane, where the x-axis is the real axis and the y-axis is the imaginary axis.

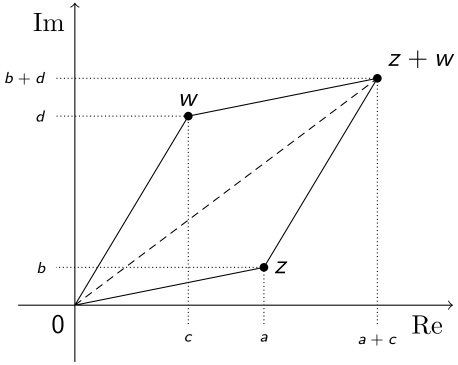


- Real numbers: $a + 0i$ lie on the real axis.
- Pure imaginary numbers: $0 + bi$ lie on the imaginary axis.

- $|z| = \sqrt{a^2 + b^2}$ is the distance from z to the origin.
- $\overline{z}$ is the reflection of z in the x-axis.

Addition

If $z = a + bi$ and $w = c + di$, then $z + w = (a + c) + (b + d)i$. Geometrically, we have:

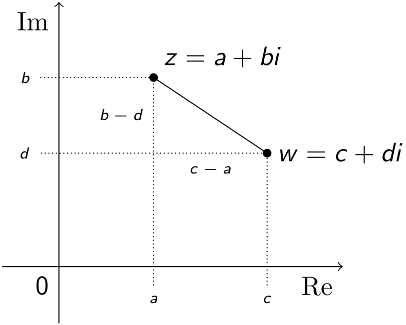


0, z, w, and $z + w$ are the vertices of a parallelogram.

Subtraction

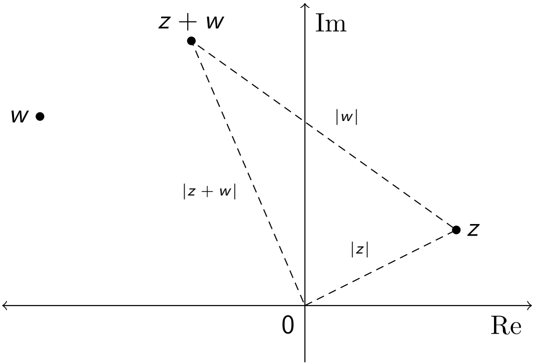
If $z = a + bi$ and $w = c + di$, then

$$|z - w| = \sqrt{(a-c)^2 + (b-d)^2}.$$



This is used to derive the triangle inequality: $|z + w| \leq |z| + |w|$.

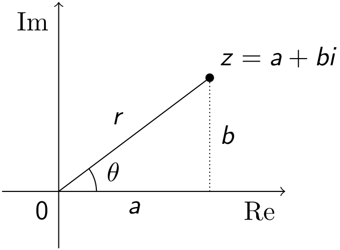
Triangle Inequality



$$|z + w| \leq |z| + |w|.$$

Polar Form

Suppose $z = a + bi$, and let $r = |z| = \sqrt{a^2 + b^2}$. Then r is the distance from z to the origin. Denote by θ the angle that the line through 0 and z makes with the positive x-axis.



Then θ is an angle defined by $\cos\theta = \frac{a}{r}$ and $\sin\theta = \frac{b}{r}$, so

$$z = r\cos\theta + r\sin\theta i = r(\cos\theta + i\sin\theta).$$

θ is called the argument of z, and is denoted $\arg z$.

Definitions

- The principal argument of $z = r(\cos\theta + i\sin\theta)$ is the angle θ such that
$$-\pi < \theta \leq \pi$$
(θ is measured in radians).
- If z is a complex number with $|z| = r$ and $\arg z = \theta$, then we write
$$z = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

Note that since $\arg z$ is not unique, $re^{i\theta}$ is a polar form of z, not the polar form of z. Adding any multiple of 2π will give another valid θ.

Examples

Convert each of the following complex numbers to polar form.

1. $3i =$
2. $-1 - i =$
3. $\sqrt{3} - i =$
4. $\sqrt{3} + 3i =$

Examples

Convert each of the following complex numbers to polar form.

1. $3i = 3e^{(\pi/2)i}$
2. $-1 - i = \sqrt{2}e^{(-3\pi/4)i}$
3. $\sqrt{3} - i = 2e^{-(\pi/6)i}$
4. $\sqrt{3} + 3i = 2\sqrt{3}e^{(\pi/3)i}$

Problems involving multiplication of complex numbers can often be simplified by using polar forms of the complex numbers.

Theorem (Appendix A, Theorem 1 – Multiplication Rule)

If $z_1 = r_1 e^{i\theta_1}$ *and* $z_2 = r_2 e^{i\theta_2}$ *are complex numbers, then*

$$z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.$$

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Theorem (Appendix A, Theorem 2 – De Moivre's Theorem)

If θ is any angle, then

$$(e^{i\theta})^n = e^{in\theta}$$

for all integers n.
(This is an obvious consequence of Theorem 1 when $n \geq 0$, but also holds when $n < 0$.)

Example

Express $(1-i)^6(\sqrt{3}+i)^3$ in the form $a+bi$.

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Solution.
Let $z = 1 - i = \sqrt{2}e^{(-\pi/4)i}$ and $w = \sqrt{3} + i = 2e^{(\pi/6)i}$. Then we want to compute $z^6 w^3$.

$$
\begin{aligned}
z^6 w^3 &= (\sqrt{2}e^{(-\pi/4)i})^6 (2e^{(\pi/6)i})^3 \\
&= (2^3 e^{(-6\pi/4)i})(2^3 e^{(3\pi/6)i}) \\
&= (8e^{(-3\pi/2)i})(8e^{(\pi/2)i}) \\
&= 64e^{-\pi i} \\
&= 64e^{\pi i} \\
&= 64(\cos\pi + i\sin\pi) \\
&= -64.
\end{aligned}
$$

Example

Express $\left(\frac{1}{2} - \frac{\sqrt{3}}{2}i\right)^{17}$ in the form $a + bi$.

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Solution.
Let $z = \frac{1}{2} - \frac{\sqrt{3}}{2}i = e^{(-\pi/3)i}$.
Then

$$\begin{aligned} z^{17} &= \left(e^{(-\pi/3)i}\right)^{17} \\ &= e^{(-17\pi/3)i} \\ &= e^{(\pi/3)i} \\ &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} \\ &= \frac{1}{2} + \frac{\sqrt{3}}{2}i. \end{aligned}$$

Roots of Unity

Example

Find **all** complex number z so that $z^3 = 1$, i.e., find the cube roots of unity. Express each root in the form $a + bi$.

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Find **all** complex number z so that $z^3 = 1$, i.e., find the cube roots of unity. Express each root in the form $a + bi$.

Let $z = re^{i\theta}$. Since $1 = 1e^{i0}$ in polar form, we want to solve

$$\left(re^{i\theta}\right)^3 = 1e^{i0},$$

i.e.,

$$r^3 e^{i3\theta} = 1e^{i0}.$$

Thus $r^3 = 1$ and $3\theta = 0 + 2\pi k = 2\pi k$ for some integer k.
Since $r^3 = 1$ and r is real, $r = 1$.

Example (continued)

Now $3\theta = 2\pi k$, so $\theta = \frac{2\pi}{3}k$.

k	θ	$e^{i\theta}$	
-3	-2π	$e^{-2\pi i}$	$= 1$
-2	$-\frac{4}{3}\pi$	$e^{(-4\pi/3)i}$	$= e^{(2\pi/3)i}$
-1	$-\frac{2}{3}\pi$	$e^{(-2\pi/3)i}$	$= e^{(-2\pi/3)i}$
0	0	e^{0i}	$= 1$
1	$\frac{2}{3}\pi$	$e^{(2\pi/3)i}$	$= e^{(2\pi/3)i}$
2	$\frac{4}{3}\pi$	$e^{(4\pi/3)i}$	$= e^{(-2\pi/3)i}$
3	2π	$e^{2\pi i}$	$= 1$

The three cube roots of unity are

$$\begin{aligned} e^{0\pi i} & & &= 1 \\ e^{(2\pi/3)i} &= \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} & &= -\frac{1}{2} + \frac{\sqrt{3}}{2}i \\ e^{(-2\pi/3)i} &= \cos\frac{-2\pi}{3} + i\sin\frac{-2\pi}{3} & &= -\frac{1}{2} - \frac{\sqrt{3}}{2}i \end{aligned}$$

Theorem (Appendix A, Theorem 3 – n^{th} Roots of Unity)

For $n \geq 1$, the (complex) solutions to $z^n = 1$ are

$$z = e^{(2\pi k/n)i} \text{ for } k = 0, 1, 2, \ldots, n-1.$$

For example, the sixth roots of unity are

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For example, the sixth roots of unity are

$$z = e^{(2\pi k/6)i} = e^{(\pi k/3)i} \text{ for } k = 0, 1, 2, 3, 4, 5.$$

k	z	
0	e^{0i}	$= 1$
1	$e^{(\pi/3)i}$	$= \frac{1}{2} + \frac{\sqrt{3}}{2}i$
2	$e^{(2\pi/3)i}$	$= -\frac{1}{2} + \frac{\sqrt{3}}{2}i$
3	$e^{\pi i}$	$= -1$
4	$e^{(4\pi/3)i}$	$= -\frac{1}{2} - \frac{\sqrt{3}}{2}i$
5	$e^{(5\pi/3)i}$	$= \frac{1}{2} - \frac{\sqrt{3}}{2}i$

Summary

1. Properties of Absolute Value
2. The Complex Plane
3. Polar Form
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