Recall

1. Complex Numbers: Basic Definitions
2. Arithmetic with Complex Numbers
3. Conjugates and Division
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Suppose that \( z = a + bi \) is a complex number.

- The **conjugate** of \( z \) is the complex number
  \[
  \bar{z} = a - bi.
  \]

- The **absolute value** or **modulus** of \( z \) is
  \[
  |z| = \sqrt{a^2 + b^2}.
  \]
Let $z$ and $w$ be complex numbers.

C1. $\overline{z \pm w} = \overline{z} \pm \overline{w}$.
C2. $\overline{zw} = \overline{z} \overline{w}$.
C3. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.
C4. $\overline{(z)} = z$.
C5. $z$ is real if and only if $\overline{z} = z$.
C6. $z \cdot \overline{z} = |z|^2$.
C7. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.
C8. $|z| \geq 0$ for all complex numbers $z$.
C9. $|z| = 0$ if and only if $z = 0$.
C10. $|zw| = |z| \cdot |w|$.
C11. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.
C12. **Triangle Inequality** $|z + w| \leq |z| + |w|$.
The Complex Plane

Represent $z = a + bi$ as a point $(a, b)$ in the plane, where the $x$-axis is the real axis and the $y$-axis is the imaginary axis.

- Real numbers: $a + 0i$ lie on the real axis.
- Pure imaginary numbers: $0 + bi$ lie on the imaginary axis.

- $|z| = \sqrt{a^2 + b^2}$ is the distance from $z$ to the origin.
- $\bar{z}$ is the reflection of $z$ in the $x$-axis.
Addition

If $z = a + bi$ and $w = c + di$, then $z + w = (a + c) + (b + d)i$. Geometrically, we have:

0, $z$, $w$, and $z + w$ are the vertices of a parallelogram.
If $z = a + bi$ and $w = c + di$, then

$$|z - w| = \sqrt{(a - c)^2 + (b - d)^2}.$$ 

This is used to derive the **triangle inequality**: $|z + w| \leq |z| + |w|$. 

![Diagram showing the subtraction of two complex numbers](image-url)
Triangle Inequality

\[ |z + w| \leq |z| + |w|. \]
Suppose $z = a + bi$, and let $r = |z| = \sqrt{a^2 + b^2}$. Then $r$ is the distance from $z$ to the origin. Denote by $\theta$ the angle that the line through 0 and $z$ makes with the positive $x$-axis.

Then $\theta$ is an angle defined by $\cos \theta = \frac{a}{r}$ and $\sin \theta = \frac{b}{r}$, so

$$z = r \cos \theta + r \sin \theta i = r(\cos \theta + i \sin \theta).$$

$\theta$ is called the argument of $z$, and is denoted $\text{arg } z$. 
The principal argument of \( z = r(\cos \theta + i \sin \theta) \) is the angle \( \theta \) such that
\[-\pi < \theta \leq \pi\] (\( \theta \) is measured in radians).

If \( z \) is a complex number with \( |z| = r \) and \( \arg z = \theta \), then we write
\[z = re^{i\theta} = r(\cos \theta + i \sin \theta).\]

Note that since \( \arg z \) is not unique, \( re^{i\theta} \) is a polar form of \( z \), not the polar form of \( z \). Adding any multiple of \( 2\pi \) will give another valid \( \theta \).
Examples

Convert each of the following complex numbers to polar form.

1. $3i = \sqrt{3} e^{\frac{\pi}{2}i}$
2. $-1 - i = \sqrt{2} e^{-\frac{3\pi}{4}i}$
3. $\sqrt{3} - i = 2 e^{-\frac{\pi}{6}i}$
4. $\sqrt{3} + 3i = 2\sqrt{3} e^{\frac{\pi}{3}i}$
Examples

Convert each of the following complex numbers to polar form.

1. $3i = 3e^{(\pi/2)i}$
2. $-1 - i = \sqrt{2}e^{(-3\pi/4)i}$
3. $\sqrt{3} - i = 2e^{-(\pi/6)i}$
4. $\sqrt{3} + 3i = 2\sqrt{3}e^{(\pi/3)i}$
Problems involving multiplication of complex numbers can often be simplified by using polar forms of the complex numbers.

**Theorem (Appendix A, Theorem 1 – Multiplication Rule)**

If \( z_1 = r_1 e^{i\theta_1} \) and \( z_2 = r_2 e^{i\theta_2} \) are complex numbers, then

\[
z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.
\]
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\[
z_1 z_2 = r_1 r_2 e^{i(\theta_1 + \theta_2)}.
\]

**Theorem (Appendix A, Theorem 2 – De Moivre’s Theorem)**

If \( \theta \) is any angle, then

\[
(e^{i\theta})^n = e^{in\theta}
\]

for all integers \( n \).

(This is an obvious consequence of Theorem 1 when \( n \geq 0 \), but also holds when \( n < 0 \).)
Example

Express \((1 - i)^6(\sqrt{3} + i)^3\) in the form \(a + bi\).
Example

Express \((1 - i)^6(\sqrt{3} + i)^3\) in the form \(a + bi\).

Solution.
Let \(z = 1 - i = \sqrt{2}e^{(-\pi/4)i}\) and \(w = \sqrt{3} + i = 2e^{(\pi/6)i}\). Then we want to compute \(z^6 w^3\).

\[
z^6 w^3 = (\sqrt{2}e^{(-\pi/4)i})^6 (2e^{(\pi/6)i})^3 \\
= (2^3 e^{(-6\pi/4)i})(2^3 e^{(3\pi/6)i}) \\
= (8e^{(-3\pi/2)i})(8e^{(\pi/2)i}) \\
= 64e^{-\pi i} \\
= 64e^{\pi i} \\
= 64(\cos \pi + i \sin \pi) \\
= -64.
\]
Example

Express \( \left( \frac{1}{2} - \frac{\sqrt{3}}{2}i \right)^{17} \) in the form \( a + bi \).
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Express \( \left( \frac{1}{2} - \frac{\sqrt{3}}{2} i \right)^{17} \) in the form \( a + bi \).

Solution.
Let \( z = \frac{1}{2} - \frac{\sqrt{3}}{2} i = e^{(-\pi/3)i} \).

Then

\[
\begin{align*}
z^{17} &= \left( e^{(-\pi/3)i} \right)^{17} \\
&= e^{(-17\pi/3)i} \\
&= e^{(\pi/3)i} \\
&= \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \\
&= \frac{1}{2} + \frac{\sqrt{3}}{2}i.
\end{align*}
\]
Roots of Unity

Example

Find all complex number $z$ so that $z^3 = 1$, i.e., find the cube roots of unity. Express each root in the form $a + bi$. 
Roots of Unity

Example

Find all complex number $z$ so that $z^3 = 1$, i.e., find the cube roots of unity. Express each root in the form $a + bi$.

Let $z = re^{i\theta}$. Since $1 = 1e^{i0}$ in polar form, we want to solve

$$\left(re^{i\theta}\right)^3 = 1e^{i0},$$

i.e.,

$$r^3 e^{i3\theta} = 1e^{i0}.$$  

Thus $r^3 = 1$ and $3\theta = 0 + 2\pi k = 2\pi k$ for some integer $k$. Since $r^3 = 1$ and $r$ is real, $r = 1$. 
Example (continued)

Now $3\theta = 2\pi k$, so $\theta = \frac{2\pi}{3} k$.

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\theta$</th>
<th>$e^{i\theta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$-3$</td>
<td>$-2\pi$</td>
<td>$e^{-2\pi i}$ = 1</td>
</tr>
<tr>
<td>$-2$</td>
<td>$-\frac{4}{3}\pi$</td>
<td>$e^{(-4\pi/3)i} = e^{(2\pi/3)i}$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$-\frac{2}{3}\pi$</td>
<td>$e^{(-2\pi/3)i} = e^{(-2\pi/3)i}$</td>
</tr>
<tr>
<td>$0$</td>
<td>$0$</td>
<td>$e^{0i} = 1$</td>
</tr>
<tr>
<td>$1$</td>
<td>$\frac{2}{3}\pi$</td>
<td>$e^{(2\pi/3)i} = e^{(2\pi/3)i}$</td>
</tr>
<tr>
<td>$2$</td>
<td>$\frac{4}{3}\pi$</td>
<td>$e^{(4\pi/3)i} = e^{(-2\pi/3)i}$</td>
</tr>
<tr>
<td>$3$</td>
<td>$2\pi$</td>
<td>$e^{2\pi i} = 1$</td>
</tr>
</tbody>
</table>

The three cube roots of unity are

\[
e^{0\pi i} = 1
\]
\[
e^{(2\pi/3)i} = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{\sqrt{3}}{2}i
\]
\[
e^{(-2\pi/3)i} = \cos \frac{-2\pi}{3} + i \sin \frac{-2\pi}{3} = -\frac{1}{2} - \frac{\sqrt{3}}{2}i
\]
Theorem (Appendix A, Theorem 3 – $n^{th}$ Roots of Unity)

For $n \geq 1$, the (complex) solutions to $z^n = 1$ are

$$z = e^{(2\pi k/n)i} \text{ for } k = 0, 1, 2, \ldots, n - 1.$$ 

For example, the sixth roots of unity are
Theorem (Appendix A, Theorem 3 – $n^{th}$ Roots of Unity)

For $n \geq 1$, the (complex) solutions to $z^n = 1$ are

$$z = e^{(2\pi k/n)i} \text{ for } k = 0, 1, 2, \ldots, n - 1.$$ 

For example, the sixth roots of unity are

$$z = e^{(2\pi k/6)i} = e^{(\pi k/3)i} \text{ for } k = 0, 1, 2, 3, 4, 5.$$ 

<table>
<thead>
<tr>
<th>$k$</th>
<th>$z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$e^{0i}$</td>
</tr>
<tr>
<td>1</td>
<td>$e^{(\pi/3)i}$</td>
</tr>
<tr>
<td>2</td>
<td>$e^{(2\pi/3)i}$</td>
</tr>
<tr>
<td>3</td>
<td>$e^{\pi i}$</td>
</tr>
<tr>
<td>4</td>
<td>$e^{(4\pi/3)i}$</td>
</tr>
<tr>
<td>5</td>
<td>$e^{(5\pi/3)i}$</td>
</tr>
</tbody>
</table>

$z = 1, \quad z = \frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z = -\frac{1}{2} + \frac{\sqrt{3}}{2}i, \quad z = -1, \quad z = -\frac{1}{2} - \frac{\sqrt{3}}{2}i, \quad z = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$
Summary

1. Properties of Absolute Value
2. The Complex Plane
3. Polar Form
4. Roots of Unity