Linear Methods (Math 211)
Lecture 11 - §2.4

(with slides adapted from K. Seyffarth)

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October 2, 2013

Recall

1. The Matrix Inversion Algorithm
2. Properties of Inversion

Today

Example

True or false? If $A^3 = 4I$, then A is invertible.

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If $A^3 = 4I$, then

$$\frac{1}{4}A^3 = I,$$

so

$$(\frac{1}{4}A^2)A = I \text{ and } A(\frac{1}{4}A^2) = I.$$

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$$(\frac{1}{4}A^2)A = I \text{ and } A(\frac{1}{4}A^2) = I.$$

Therefore A is invertible, and $A^{-1} = \frac{1}{4}A^2$. **True**

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Take $A = I$ and $B = -I$. Both are invertible but $A + B = 0$ is not.
False

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Corollary

If A and C are $n \times n$ matrices such that $AC = I$, then $CA = I$ and $C = A^{-1}$, $A = C^{-1}$.

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Example

Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \text{ and } C = \begin{bmatrix} -1 & 1 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}.$$

$$AC = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad CA = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

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Therefore AB is the inverse of A. **True**

This is the end of the material to be included on the midterm

The Inverse of a Matrix Transformation

Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is a matrix transformation induced by an **invertible** matrix A, i.e.,

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Geometrically, S reverses the action of T, and T reverses the action of S, and S is called an inverse of T.

Theorem (§2.4 Theorem 6)

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a matrix transformation induced by matrix A. Then A is invertible if and only if T has an inverse. In the case where T has an inverse, the inverse is unique and is denoted T^{-1}. Furthermore, $T^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is induced by the matrix A^{-1}.

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or

1. $T^{-1} \circ T = 1_{\mathbb{R}^n}$
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Example (§2.4 Example 13)

Let $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ denote counterclockwise rotation of the plane about the origin through an angle of θ. R_θ is induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

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We can verify that this is correct by computing AA^{-1}.

Definition

An elementary matrix is a matrix obtained from an identity matrix by performing **a single** elementary row operation.

Example

$$E = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, F = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, G = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

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are examples of elementary matrices of types I, II and III, respectively. Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix}$; compute EA, FA, and GA.

Example (continued)

$$EA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 4 \\ 3 & 3 \\ 2 & 2 \end{bmatrix};$$

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$$GA = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 3 & 3 \\ 4 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 0 \\ 4 & 4 \end{bmatrix}.$$

Lemma (§2.5 Lemma 1)

If A is an $m \times n$ matrix, and B is obtained from A by performing a single elementary row operation, then $B = EA$ where E is the elementary matrix obtained by performing the same elementary operation on I_m.

Summary