Linear Methods (Math 211)
Lecture 16 - Appendix A

(with slides adapted from K. Seyffarth)

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Recall

1. More Matrix Transformations
2. Vector Operations
3. Reflections and Rotations

Today

1. Basic Definitions
2. Arithmetic with Complex Numbers
3. Conjugates and Division

Why complex numbers?

- Counting numbers: $1, 2, 3, 4, 5, \ldots$
- Integers: $0, 1, 2, 3, 4 \ldots$ but also $-1, -2, -3 \ldots$.
- To solve $3x + 2 = 0$, integers aren't enough, so we have rational numbers (fractions), e.g.,
$$\text{If } 3x + 2 = 0, \text{ then } x = -\frac{2}{3}.$$
- But there are no rational numbers x with the property that $x^2 - 2 = 0$, so we allow irrational numbers, e.g.,
$$\text{If } x^2 - 2 = 0, \text{ then } x = \pm\sqrt{2}.$$
- The set of real numbers, $\mathbb{R}$, consists of numbers that can be written as decimals, both rational and irrational. However, we still can't solve
$$x^2 = -1$$
since the square of a positive number and of a negative number are both positive.

Complex Numbers

- The imaginary unit, denoted i, is defined to be a number with the property that $i^2 = -1$.
- A pure imaginary number has the form bi where $b \in \mathbb{R}$, $b \neq 0$, and i is the imaginary unit.
- A complex number is any number z of the form

$$z = a + bi$$

where $a, b \in \mathbb{R}$ and i is the imaginary unit.

  - a is called the real part of z.
  - b is called the imaginary part of z.
  - If $b = 0$, then z is a real number.

Operations with Complex Numbers

Let $z = a + bi$ and $w = c + di$ be complex numbers.

- Equality. $z = w$ if and $a = c$ and $b = d$.
- Addition and Subtraction.

$$z + w = (a + bi) + (c + di) = (a + c) + (b + d)i$$
$$z - w = (a + bi) - (c + di) = (a - c) + (b - d)i$$

- Multiplication.

$$zw = (a + bi)(c + di) = (ac - bd) + (ad + bc)i$$

Examples

$$(-3+6i)+(5-i)=$$
$$(4-7i)-(6-2i)=$$
$$(2-3i)(-3+4i)=$$

Examples

$$(-3 + 6i) + (5 - i) = 2 + 5i.$$
$$(4 - 7i) - (6 - 2i) = -2 - 5i.$$
$$(2 - 3i)(-3 + 4i) = -6 + 8i + 9i + 12 = 6 + 17i.$$

Example

Find all complex number $z = (a + bi)$ so that $z^2 = -3 + 4i$.

Example

Find all complex number $z = (a + bi)$ so that $z^2 = -3 + 4i$.

$$z^2 = (a + bi)^2 = (a^2 - b^2) + 2abi = -3 + 4i,$$

so

$$a^2 - b^2 = -3 \text{ and } 2ab = 4.$$

Since $2ab = 4$, $a = \frac{2}{b}$. Substitute this into the first equation:

$$\begin{aligned} a^2 - b^2 &= -3 \\ (2/b)^2 - b^2 &= -3 \\ \frac{4}{b^2} - b^2 &= -3 \\ 4 - b^4 &= -3b^2 \\ b^4 - 3b^2 - 4 &= 0. \end{aligned}$$

Example (continued)

Now, $b^4 - 3b^2 - 4 = 0$ can be factored into

$$(b^2 - 4)(b^2 + 1) = 0$$
$$(b - 2)(b + 2)(b^2 + 1) = 0.$$

Since $b \in \mathbb{R}$, and $b^2 + 1$ has no real roots, $b = 2$ or $b = -2$.

Since $a = \frac{2}{b}$, it follows that

- when $b = 2$, $a = 1$, and $z = a + bi = 1 + 2i$;
- when $b = -2$, $a = -1$, and $z = a + bi = -1 - 2i$.

Therefore, if $z^2 = -3 + 4i$, then $z = 1 + 2i$ or $z = -1 - 2i$.

Definitions

Let $z = a + bi$ and $w = c + di$ be complex numbers.

- The conjugate of z is the complex number

$$\bar{z} = a - bi.$$

- Division. Suppose that c, d are not both zero. Then

$$\begin{aligned}\frac{a+bi}{c+di} &= \frac{a+bi}{c+di} \times \frac{c-di}{c-di} \\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2} \\ &= \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i.\end{aligned}$$

Examples

$$\frac{1}{i} =$$

$$\frac{1}{3+4i} =$$

$$\frac{1-2i}{-2+5i} =$$

Examples

$$\frac{1}{i} = \frac{1}{i} \times \frac{-i}{-i} = \frac{-i}{-i^2} = -i.$$

$$\begin{aligned}
\frac{1}{3+4i} &= \frac{1}{3+4i} \times \frac{3-4i}{3-4i} \\
&= \frac{3-4i}{3^3+4^2} \\
&= \frac{3}{25} - \frac{4}{25}i. \\
\frac{1-2i}{-2+5i} &= \frac{1-2i}{-2+5i} \times \frac{-2-5i}{-2-5i} \\
&= \frac{(-2-10)+(4-5)i}{2^2+5^2} \\
&= -\frac{12}{29} - \frac{1}{29}i.
\end{aligned}$$

Definition

The absolute value or modulus of a complex number $z = a + bi$ is

$$|z| = \sqrt{a^2 + b^2}.$$

Note that this is consistent with the definition of the absolute value of a real number.

Examples

$$|-3 + 4i| =$$

$$|3 - 2i| =$$

$$|i| =$$

Definition

The absolute value or modulus of a complex number $z = a + bi$ is

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Examples

$$|-3 + 4i| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5.$$

$$|3 - 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}.$$

$$|i| = \sqrt{1^2} = 1.$$

Properties of the Conjugate and Absolute Value (p. 507)

Let z and w be complex numbers.

C1. $\overline{z \pm w} = \overline{z} \pm \overline{w}$.

C2. $\overline{(zw)} = \overline{z}\ \overline{w}$.

C3. $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$.

C4. $\overline{(\overline{z})} = z$.

C5. z is real if and only if $\overline{z} = z$.

C6. $z \cdot \overline{z} = |z|^2$.

C7. $\frac{1}{z} = \frac{\overline{z}}{|z|^2}$.

C8. $|z| \geq 0$ for all complex numbers z

C9. $|z| = 0$ if and only if $z = 0$.

C10. $|zw| = |z|\ |w|$.

C11. $\left|\frac{z}{w}\right| = \frac{|z|}{|w|}$.

C12. **Triangle Inequality** $|z + w| \leq |z| + |w|$.

Summary

1. Basic Definitions
2. Arithmetic with Complex Numbers
3. Conjugates and Division