Linear Methods (Math 211)
Lecture 15 - §2.9

(with slides adapted from K. Seyffarth)

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October 16, 2013

§2.9 Markov Chains

Markov Chains are used to model systems (or processes) that evolve through a series of stages. At each stage, the system is in one of a finite number of states.

Example (Weather Model)

Three states: sunny (S), cloudy (C), rainy (R).
Stages: days.

The state that the system occupies at any stage is determined by a set of probabilities.

Important fact: probabilities are always real numbers between zero and one, inclusive.

Example (Weather Model – continued)

- If it is sunny one day, then there is a 40% chance it will be sunny the next day, and a 40% chance that it will be cloudy the next day (and a 20% chance it will be rainy the next day).

The values 40%, 40% and 20% are transition probabilities, and are assumed to be known.

- If it is cloudy one day, then there is a 40% chance it will be rainy the next day, and a 25% chance that it will be sunny the next day.
- If it is rainy one day, then there is a 30% chance it will be rainy the next day, and a 50% chance that it will be cloudy the next day.

Example (Weather Model – continued)

We put the transition probabilities into a transition matrix,

$$P = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix}$$

Note. Transition matrices are stochastic, meaning that the sum of the entries in each column is equal to one.

Suppose that it is rainy on Thursday. What is the probability that it will be sunny on Sunday?

The initial state vector, S_0, corresponds to the state of the weather on Thursday, so

$$S_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Example (Weather Model – continued)

What is the state vector for Friday?

$$S_1 = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = PS_0.$$

To find the state vector for Saturday:

$$S_2 = PS_1 = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.265 \\ 0.405 \\ 0.33 \end{bmatrix}$$

Finally, the state vector for Sunday is

$$S_3 = PS_2 = \begin{bmatrix} 0.4 & 0.25 & 0.2 \\ 0.4 & 0.35 & 0.5 \\ 0.2 & 0.4 & 0.3 \end{bmatrix} \begin{bmatrix} 0.265 \\ 0.405 \\ 0.33 \end{bmatrix} = \begin{bmatrix} 0.27325 \\ 0.41275 \\ 0.314 \end{bmatrix}$$

The probability that it will be sunny on Sunday is 27.325%.

Important fact: the sum of the entries of a state vector is always one.

Theorem (§2.9 Theorem 1)

If P is the transition matrix for an n-state Markov chain, then

$$S_{m+1} = PS_m \text{ for } m = 0, 1, 2, \ldots$$

Example (§2.9 Example 1)

- A customer always eats lunch either at restaurant A or restaurant B.
- The customer never eats at A two days in a row.
- If the customer eats at B one day, then the next day she is three times as likely to eat at B as at A.

What is the probability transition matrix?

$$P = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$$

Example (continued)

Initially, the customer is equally likely to eat at either restaurant, so

$$S_0 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$$

$$S_1 = \begin{bmatrix} 0.125 \\ 0.875 \end{bmatrix}, S_2 = \begin{bmatrix} 0.21875 \\ 0.78125 \end{bmatrix}, S_3 = \begin{bmatrix} 0.1953125 \\ 0.8046875 \end{bmatrix},$$

$$S_4 = \begin{bmatrix} 0.20117 \\ 0.79883 \end{bmatrix}, S_5 = \begin{bmatrix} 0.19971 \\ 0.80029 \end{bmatrix},$$

$$S_6 = \begin{bmatrix} 0.20007 \\ 0.79993 \end{bmatrix}, S_7 = \begin{bmatrix} 0.19998 \\ 0.80002 \end{bmatrix},$$

are calculated, and these appear to converge to

$$\begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

Example (§2.9 Example 3)

A wolf pack always hunts in one of three regions, R_1, R_2, and R_3.

- If it hunts in some region one day, it is as likely as not to hunt there again the next day.
- If it hunts in R_1, it never hunts in R_2 the next day.
- If it hunts in R_2 or R_3, it is equally likely to hunt in each of the other two regions the next day.

If the pack hunts in R_1 on Monday, find the probability that it will hunt in R_3 on Friday.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \text{ and } S_0 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We want to find S_4, and, in particular, the last entry in S_4.

Example (continued)

$$S_1 = \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix}, S_2 = PS_1 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ 0 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{1}{2} \end{bmatrix},$$

$$S_3 = PS_2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{3}{8} \\ \frac{1}{8} \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{11}{32} \\ \frac{3}{16} \\ \frac{15}{32} \end{bmatrix},$$

$$S_4 = PS_3 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{11}{32} \\ \frac{3}{16} \\ \frac{15}{32} \end{bmatrix} = \begin{bmatrix} \\ \\ \frac{29}{64} \end{bmatrix}$$

Therefore, the probability of the pack hunting in R_3 on Friday is $\frac{29}{64}$.

Steady State Vectors

Sometimes, state vectors converge to a particular vector, called the steady state vector.

Problem

How do we know if a Markov chain has a steady state vector? If the Markov chain has a steady state vector, how do we find it?

One condition ensuring that a steady state vector exists is that the transition matrix P be regular, meaning that for some integer $k > 0$, all entries of P^k are positive (i.e., greater than zero).

In §2.9 Example 1, $P = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix}$ is regular because

$$P^2 = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} \frac{1}{4} & \frac{3}{16} \\ \frac{3}{4} & \frac{13}{16} \end{bmatrix}$$

has all entries greater than zero.

Theorem (§2.9 Theorem 2 – paraphrased)

If P is the transition matrix of a Markov chain and P is regular, then the steady state vector can be found by solving the system

$$S = PS$$

for S, and then ensuring that the entries of S sum to one.

Notice that if $S = PS$, then

$$\begin{aligned} S - PS &= 0 \\ IS - PS &= 0 \\ (I - P)S &= 0 \end{aligned}$$

- This last line represents a system of linear equations that is homogeneous.
- The structure of P ensures that $I - P$ is not invertible, and so the system has infinitely many solutions.
- Choose the value of the parameter so that the entries of S sum to one.

Example

From §2.9 Example 1,

$$P = \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix},$$

and we've already verified that P is regular.

Now solve the system $(I-P)S = 0$.

$$I - P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{4} \\ 1 & \frac{3}{4} \end{bmatrix} = \begin{bmatrix} 1 & -\frac{1}{4} \\ -1 & \frac{1}{4} \end{bmatrix}$$

Solving $(I-P)S = 0$:

$$\left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ -1 & \frac{1}{4} & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -\frac{1}{4} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution in parametric form is

$$s_1 = \frac{1}{4}t, s_2 = t \text{ for } t \in \mathbb{R}.$$

Example (continued)

Since $s_1 + s_2 = 1$,

$$\begin{aligned} \frac{1}{4}t + t &= 1 \\ \frac{5}{4}t &= 1 \\ t &= \frac{4}{5} \end{aligned}$$

Therefore, the steady state vector is

$$S = \begin{bmatrix} \frac{1}{5} \\ \frac{4}{5} \end{bmatrix} = \begin{bmatrix} 0.2 \\ 0.8 \end{bmatrix}$$

Example (§2.9 Example 3)

Is there a steady state vector? If so, find it.

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

so

$$P^2 = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \\ 0 & \frac{1}{2} & \frac{1}{4} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{2} \end{bmatrix} = \begin{bmatrix} \frac{5}{8} & \frac{5}{16} & \frac{5}{16} \\ \frac{1}{8} & \frac{5}{16} & \frac{1}{4} \\ \frac{1}{2} & \frac{3}{8} & \frac{7}{16} \end{bmatrix}$$

Therefore P is regular, and there is definitely a steady state by Theorem 2.

Example (continued)

Now solve the system $(I-P)S=0$.

$$\left[\begin{array}{rrr|r} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ -\frac{1}{2} & -\frac{1}{4} & \frac{1}{2} & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} \frac{1}{2} & -\frac{1}{4} & -\frac{1}{4} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{4} & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & -\frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{3}{4} & 0 \\ 0 & \frac{1}{2} & -\frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \left[\begin{array}{rrr|r} 1 & 0 & -\frac{3}{4} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution in parametric form is

$$s_3 = t, s_2 = \frac{1}{2}t, s_1 = \frac{3}{4}t, \text{ where } t \in \mathbb{R}.$$

Example (continued)

Since $s_1 + s_2 + s_3 = 1$,

$$t + \frac{1}{2}t + \frac{3}{4}t = 1,$$

implying that $t = \frac{4}{9}$. Therefore the steady state vector is

$$S = \begin{bmatrix} \frac{3}{9} \\ \frac{2}{9} \\ \frac{4}{9} \end{bmatrix}.$$