Linear Methods (Math 211)
Lecture 14 - §2.6

(with slides adapted from K. Seyffarth)

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Recall

1. Products of Elementary Matrices
2. Linear Transformations
3. ... and Matrix Transformations

Today

1. More Matrix Transformations
2. Vector Operations
3. Reflections and Rotations

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be a transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix}.$$

Show that T is a linear transformation.

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Show that T is a linear transformation.

Solution. If T were a linear transformation, then T would be induced by the matrix

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix} = \begin{bmatrix} T \begin{bmatrix} 1 \\ 0 \end{bmatrix} & T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}.$$

Example (continued)

Since

$$A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x+2y \end{bmatrix} = T\begin{bmatrix} x \\ y \end{bmatrix},$$

T is a matrix transformation, and therefore a linear transformation.

Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x + y \end{bmatrix}.$$

Is T is a linear transformation? Explain.

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Now

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x+y \end{bmatrix}.$$

Example (continued)

In particular,

$$A\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}0\\2\end{bmatrix},$$

while

$$T\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\2\end{bmatrix}.$$

Since $A\begin{bmatrix}1\\1\end{bmatrix} \neq T\begin{bmatrix}1\\1\end{bmatrix}$, T is **not** a linear transformation.

There is an alternate way to show that T is not a linear transformation.

Example (continued)

Notice that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

From this we see that

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + T \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

i.e., T does not preserve addition, and so T is **not** a linear transformation.

Geometric Interpretation of Vector Operations

Definition (Vector scalar multiplication)

Let $\mathbf{x} \in \mathbb{R}^2$ and let $k \in \mathbb{R}$. Then $k\mathbf{x}$ is the vector in $\mathbb{R}^2$ that is $|k|$ times the length of $\mathbf{x}$; $k\mathbf{x}$ points the same directions as $\mathbf{x}$ if $k > 0$, and opposite to $\mathbf{x}$ if $k < 0$.

Definition (Vector addition)

Let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$, and consider the parallelogram defined by 0, $\mathbf{x}$ and $\mathbf{y}$. The vector $\mathbf{x} + \mathbf{y}$ corresponds to the fourth vertex of this parallelogram.

Reflection in $\mathbb{R}^2$

Let $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ denote reflection across the line $y = mx$, and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $k \in \mathbb{R}$. Q_m is a linear transformation and hence a matrix transformation.

Example

Find the matrix inducing Q_m.

Reflection in $\mathbb{R}^2$

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Example

Find the matrix inducing Q_m.

By Theorem 2, we can find the matrix that induces Q_m by finding $Q_m(\mathbf{e}_1)$ and $Q_m(\mathbf{e}_2)$.

Example (continued)

However, an easier way to obtain the matrix for Q_m is to use the following observation:

$$Q_m = R_\theta \circ Q_0 \circ R_{-\theta},$$

where θ is the angle between $y = mx$ and the positive x axis.

Using the standard trigonometric identities:

$$\cos\theta = \frac{1}{\sqrt{1+m^2}} \text{ and } \sin\theta = \frac{m}{\sqrt{1+m^2}},$$

the matrix for Q_m can be found by computing the product

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{bmatrix}$$

Theorem (§2.6 Theorem 5)

The transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, denoting reflection across the line $y = mx$, is a linear transformation and is induced by the matrix

$$\frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

For reflection across the x-axis, $m = 0$, and the theorem yields the expected matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The y-axis has undefined slope, so the theorem does not apply. We use Q_Y to denote reflection across the y-axis; Q_Y is induced by the matrix

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Example

Find the rotation or reflection that equals reflection across the x-axis followed by rotation through an angle of $\frac{\pi}{2}$.

Example

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We must find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

Q_0 is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$R_{\frac{\pi}{2}} \circ Q_0$ is induced by

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Example (continued)

We've seen this matrix before: $BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ is a reflection matrix.

In fact,

$$R_{\frac{\pi}{2}} \circ Q_0 = Q_1,$$

reflection across the line $y = x$.

In general,

- The composite of two reflections is a rotation.
- The composite of two rotations is a rotation.
- The composite of a reflection and a rotation is a reflection.

Both reflections and rotations are orthogonal: $A^{-1} = A^T$. The rotations are the 2×2 orthogonal matrices with determinant 1, and the reflections are those with determinant -1.

Example

Find the rotation or reflection that equals reflection across the line $y = -x$ followed by reflection across the y-axis.

Example

Find the rotation or reflection that equals reflection across the line $y = -x$ followed by reflection across the y-axis.

We must find the matrix for the transformation $Q_Y \circ Q_{-1}$.

Q_{-1} is induced by

$$A = \frac{1}{2}\begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},$$

and Q_Y is induced by

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore, $Q_Y \circ Q_{-1}$ is induced by BA.

Example (continued)

$$BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

We know BA is a rotation, and it must be rotation through an angle θ such that

$$\cos\theta = 0 \text{ and } \sin\theta = -1.$$

Therefore, $Q_Y \circ Q_{-1} = R_{-\frac{\pi}{2}} = R_{\frac{3\pi}{2}}$.

Summary

1. More Matrix Transformations
2. Vector Operations
3. Reflections and Rotations