Recall

1. Products of Elementary Matrices
2. Linear Transformations
3. ... and Matrix Transformations
Today

1. More Matrix Transformations
2. Vector Operations
3. Reflections and Rotations
Example

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^3 \) be a transformation defined by

\[
T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix}.
\]

Show that \( T \) is a linear transformation.
Example

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$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix}.$$ 

Show that $T$ is a linear transformation.

**Solution.** If $T$ were a linear transformation, then $T$ would be induced by the matrix

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} T \begin{bmatrix} 1 \\ 0 \end{bmatrix} & T \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix}.$$
Example (continued)

Since

\[
A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2x \\ y \\ -x + 2y \end{bmatrix} = T \begin{bmatrix} x \\ y \end{bmatrix},
\]

\(T\) is a matrix transformation, and therefore a linear transformation.
Example

Let \( T : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) be a transformation defined by

\[
T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x + y \end{bmatrix}.
\]

Is \( T \) is a linear transformation? Explain.
Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x + y \end{bmatrix}.$$ 

Is $T$ is a linear transformation? Explain.

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Example

Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a transformation defined by

$$T \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} xy \\ x + y \end{bmatrix}.$$ 

Is $T$ is a linear transformation? Explain.

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Now

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x + y \end{bmatrix}.$$
Example (continued)

In particular,

\[
A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix},
\]

while

\[
T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Since \(A \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq T \begin{bmatrix} 1 \\ 1 \end{bmatrix}\), \(T\) is **not** a linear transformation.

There is an alternate way to show that \(T\) is not a linear transformation.
Example (continued)

Notice that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad T \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad T \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$  

From this we see that

$$T \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq T \begin{bmatrix} 1 \\ 0 \end{bmatrix} + T \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

i.e., $T$ does not preserve addition, and so $T$ is not a linear transformation.
Geometric Interpretation of Vector Operations

Definition (Vector scalar multiplication)

Let \( \mathbf{x} \in \mathbb{R}^2 \) and let \( k \in \mathbb{R} \). Then \( k\mathbf{x} \) is the vector in \( \mathbb{R}^2 \) that is \( |k| \) times the length of \( \mathbf{x} \); \( k\mathbf{x} \) points the same directions as \( \mathbf{x} \) if \( k > 0 \), and opposite to \( \mathbf{x} \) if \( k < 0 \).

Definition (Vector addition)

Let \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^2 \), and consider the parallelogram defined by 0, \( \mathbf{x} \) and \( \mathbf{y} \). The vector \( \mathbf{x} + \mathbf{y} \) corresponds to the fourth vertex of this parallelogram.
Reflection in $\mathbb{R}^2$

Let $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection across the line $y = mx$, and let $\mathbf{x}, \mathbf{y} \in \mathbb{R}^2$ and $k \in \mathbb{R}$. $Q_m$ is a linear transformation and hence a matrix transformation.

**Example**

Find the matrix inducing $Q_m$. 
Reflection in $\mathbb{R}^2$

Let $Q_m : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ denote reflection across the line $y = mx$, and let $x, y \in \mathbb{R}^2$ and $k \in \mathbb{R}$. $Q_m$ is a linear transformation and hence a matrix transformation.

Example

Find the matrix inducing $Q_m$.

By Theorem 2, we can find the matrix that induces $Q_m$ by finding $Q_m(e_1)$ and $Q_m(e_2)$. 
Example (continued)

However, an easier way to obtain the matrix for $Q_m$ is to use the following observation:

$$Q_m = R_\theta \circ Q_0 \circ R_{-\theta},$$

where $\theta$ is the angle between $y = mx$ and the positive $x$ axis.

Using the standard trigonometric identities:

$$\cos \theta = \frac{1}{\sqrt{1 + m^2}} \text{ and } \sin \theta = \frac{m}{\sqrt{1 + m^2}},$$

the matrix for $Q_m$ can be found by computing the product

$$\begin{bmatrix}
\cos \theta & -\sin \theta \\
\sin \theta & \cos \theta
\end{bmatrix}
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}
\begin{bmatrix}
\cos(-\theta) & -\sin(-\theta) \\
\sin(-\theta) & \cos(-\theta)
\end{bmatrix}.$$
The transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$, denoting reflection across the line $y = mx$, is a linear transformation and is induced by the matrix

$$
\frac{1}{1 + m^2} \begin{bmatrix}
1 - m^2 & 2m \\
2m & m^2 - 1
\end{bmatrix}.
$$

For reflection across the $x$-axis, $m = 0$, and the theorem yields the expected matrix

$$
\begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
$$

The $y$-axis has undefined slope, so the theorem does not apply. We use $Q_Y$ to denote reflection across the $y$-axis; $Q_Y$ is induced by the matrix

$$
\begin{bmatrix}
-1 & 0 \\
0 & 1
\end{bmatrix}.
Example

Find the rotation or reflection that equals reflection across the $x$-axis followed by rotation through an angle of $\frac{\pi}{2}$. 

$Q_0$ is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by $B = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. 

$R_{\frac{\pi}{2}} \circ Q_0$ is induced by $BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$. 


Example

Find the rotation or reflection that equals reflection across the $x$-axis followed by rotation through an angle of $\frac{\pi}{2}$.

We must find the matrix for the transformation $R_{\frac{\pi}{2}} \circ Q_0$.

$Q_0$ is induced by $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $R_{\frac{\pi}{2}}$ is induced by

$$B = \begin{bmatrix} \cos \frac{\pi}{2} & -\sin \frac{\pi}{2} \\ \sin \frac{\pi}{2} & \cos \frac{\pi}{2} \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$R_{\frac{\pi}{2}} \circ Q_0$ is induced by

$$BA = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$
Example (continued)

We’ve seen this matrix before: \( BA = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \) is a \textit{reflection} matrix.

In fact, \( R_{\frac{\pi}{2}} \circ Q_0 = Q_1 \),

reflection across the line \( y = x \).
In general,

- The composite of two reflections is a rotation.
- The composite of two rotations is a rotation.
- The composite of a reflection and a rotation is a reflection.

Both reflections and rotations are **orthogonal**: $A^{-1} = A^T$. The rotations are the $2 \times 2$ orthogonal matrices with determinant 1, and the reflections are those with determinant $-1$. 
Example

Find the rotation or reflection that equals reflection across the line \( y = -x \) followed by reflection across the \( y \)-axis.
Example

Find the rotation or reflection that equals reflection across the line \( y = -x \) followed by reflection across the \( y \)-axis.

We must find the matrix for the transformation \( Q_Y \circ Q_{-1} \).

\( Q_{-1} \) is induced by

\[
A = \frac{1}{2} \begin{bmatrix} 0 & -2 \\ -2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix},
\]

and \( Q_Y \) is induced by

\[
B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.
\]

Therefore, \( Q_Y \circ Q_{-1} \) is induced by \( BA \).
Example (continued)

\[ BA = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}. \]

We know \( BA \) is a rotation, and it must be rotation through an angle \( \theta \) such that

\[ \cos \theta = 0 \quad \text{and} \quad \sin \theta = -1. \]

Therefore, \( Q_Y \circ Q_{-1} = R_{-\frac{\pi}{2}} = R_{\frac{3\pi}{2}}. \)
Summary

1. More Matrix Transformations
2. Vector Operations
3. Reflections and Rotations