# Linear Methods (Math 211)
# Lecture 25 - §3.3

(with slides adapted from K. Seyffarth)

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# Recall

1. Polynomial Interpolation
2. Vandermonde Determinants
3. Diagonalization

# Today

# Eigenvalues and Eigenvectors

## Definition

Let $A$ be an $n \times n$ matrix, $\lambda$ a real number, and $\mathbf{x} \neq 0$ an $n$-vector. If $A\mathbf{x} = \lambda\mathbf{x}$, then $\lambda$ is an eigenvalue of $A$, and $\mathbf{x}$ is an eigenvector of $A$ corresponding to $\lambda$, or a $\lambda$-eigenvector.

## Examples

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{x}.$$

This means that 3 is an eigenvalue of $A$, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A$ corresponding to 3 (or a 3-eigenvector of $A$).

# Finding all Eigenvalues and Eigenvectors of a Matrix

Suppose that $A$ is an $n \times n$ matrix, $\mathbf{x} \neq 0$ an $n$-vector, $\lambda \in \mathbb{R}$, and that $A\mathbf{x} = \lambda\mathbf{x}$. Then

$$\begin{aligned} \lambda\mathbf{x} - A\mathbf{x} &= 0 \\ \lambda I\mathbf{x} - A\mathbf{x} &= 0 \\ (\lambda I - A)\mathbf{x} &= 0 \end{aligned}$$

Since $\mathbf{x} \neq 0$, the matrix $\lambda I - A$ has no inverse, and thus

$$\det(\lambda I - A) = 0.$$

## Definition

The characteristic polynomial of an $n \times n$ matrix $A$ is

$$c_A(x) = \det(xI - A).$$

## Example

Find the characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

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Find the characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

$$\begin{aligned} c_A(x) &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= \det\begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix} \\ &= (x-4)(x-3) - 2 \\ &= x^2 - 7x + 10 \end{aligned}$$

**Theorem (§3.3 Theorem 2)**

*Let $A$ be an $n \times n$ matrix.*

1. *The eigenvalues of $A$ are the roots of $c_A(x)$.*
2. *The $\lambda$-eigenvectors $\mathbf{x}$ are the nontrivial solutions to $(\lambda I - A)\mathbf{x} = 0$.*

**Example (continued)**

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## Example (continued)

Find the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

We have

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so $A$ has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.

# Eigenvectors

**Example (continued)**

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# Eigenvectors

## Example (continued)

Find the eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

To find the 2-eigenvectors of $A$, solve $(2I - A)\mathbf{x} = 0$:

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution, in parametric form, is

$$\mathbf{x} = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

## Example (continued)

Recall that $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

To find the 5-eigenvectors of $A$, solve $(5I - A)\mathbf{x} = 0$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution, in parametric form, is

$$\mathbf{x} = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R}.$$

# Basic Eigenvectors

**Definition**

A basic eigenvector of an $n \times n$ matrix $A$ is any nonzero multiple of a basic solution to $(\lambda I - A)\mathbf{x} = 0$, where $\lambda$ is an eigenvalue of $A$.

**Example (continued)**

$\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ are basic eigenvectors of the matrix

$$A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

corresponding to eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$, respectively.

### Example

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of $A$, and the corresponding basic eigenvectors.

## Example

For $A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$, find $c_A(x)$, the eigenvalues of $A$, and the corresponding basic eigenvectors.

$$\det(xI - A) = \begin{vmatrix} x-3 & 4 & -2 \\ -1 & x+2 & -2 \\ -1 & 5 & x-5 \end{vmatrix} = \begin{vmatrix} x-3 & 4 & -2 \\ 0 & x-3 & -x+3 \\ -1 & 5 & x-5 \end{vmatrix}$$

$$= \begin{vmatrix} x-3 & 4 & 2 \\ 0 & x-3 & 0 \\ -1 & 5 & x \end{vmatrix} = (x-3) \begin{vmatrix} x-3 & 2 \\ -1 & x \end{vmatrix}$$

$$c_A(x) = (x-3)(x^2 - 3x + 2) = (x-3)(x-2)(x-1).$$

## Example (continued)

Therefore, the eigenvalues of $A$ are $\lambda_1 = 3, \lambda_2 = 2$, and $\lambda_3 = 1$.

Basic eigenvectors corresponding to $\lambda_1 = 3$: solve $(3I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{rrr|r} 0 & 4 & -2 & 0 \\ -1 & 5 & -2 & 0 \\ -1 & 5 & -2 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \begin{bmatrix} \frac{1}{2}t \\ \frac{1}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}$, $t \in \mathbb{R}$.

Choosing $t = 2$ gives us $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ as a basic eigenvector corresponding to $\lambda_1 = 3$.

## Example (continued)

Basic eigenvectors corresponding to $\lambda_2 = 2$: solve $(2I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{rrr|r} -1 & 4 & -2 & 0 \\ -1 & 4 & -2 & 0 \\ -1 & 5 & -3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \begin{bmatrix} 2s \\ s \\ s \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $s \in \mathbb{R}$.

Choosing $s = 1$ gives us $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ as a basic eigenvector corresponding to $\lambda_2 = 2$.

## Example (continued)

Basic eigenvectors corresponding to $\lambda_3 = 1$: solve $(I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{rrr|r} -2 & 4 & -2 & 0 \\ -1 & 3 & -2 & 0 \\ -1 & 5 & -4 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus $\mathbf{x} = \begin{bmatrix} r \\ r \\ r \end{bmatrix} = r \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $r \in \mathbb{R}$.

Choosing $r = 1$ gives us $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ as a basic eigenvector corresponding to $\lambda_3 = 1$.

# Geometric Interpretation of Eigenvalues and Eigenvectors

Let $A$ be a $2 \times 2$ matrix. Then $A$ can be interpreted as a linear transformation $T_A$ from $\mathbb{R}^2$ to $\mathbb{R}^2$.

**Problem**

*How does $T_A$ affect the eigenvectors of the matrix?*

# Geometric Interpretation of Eigenvalues and Eigenvectors

Let $A$ be a $2 \times 2$ matrix. Then $A$ can be interpreted as a linear transformation $T_A$ from $\mathbb{R}^2$ to $\mathbb{R}^2$.

## Problem

*How does $T_A$ affect the eigenvectors of the matrix?*

## Definition

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$. We denote by $L_{\mathbf{v}}$ the unique line in $\mathbb{R}^2$ that contains $\mathbf{v}$ and the origin.

## Lemma (§3.3 Lemma 1)

*Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$. Then $L_{\mathbf{v}}$ is the set of all scalar multiples of $\mathbf{v}$, i.e.*

$$L_{\mathbf{v}} = \mathbb{R}\mathbf{v} = \{t\mathbf{v} \mid t \in \mathbb{R}\}.$$

### Definition

Let $A$ be a $2 \times 2$ matrix and $L$ a line in $\mathbb{R}^2$ through the origin. Then $L$ is said to be $A$-invariant if the vector $A\mathbf{x}$ lies in $L$ whenever $\mathbf{x}$ lies in $L$:

- $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$,
- $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda \in \mathbb{R}$,
- $\mathbf{x}$ is an eigenvector of $A$.

### Theorem (§3.3 Theorem 3)

*Let $A$ be a $2 \times 2$ matrix and let $\mathbf{v} \neq 0$ be a vector in $\mathbb{R}^2$. Then $L_{\mathbf{v}}$ is $A$-invariant if and only if $\mathbf{v}$ is an eigenvector of $A$.*

This theorem provides a geometrical method for finding the eigenvectors of a $2 \times 2$ matrix.

## Example (§3.3 Example 6)

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $y = mx$. Find the eigenvalues and eigenvectors of $Q_m$.

## Example (§3.3 Example 6)

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $y = mx$. Find the eigenvalues and eigenvectors of $Q_m$.

The matrix that induces $Q_m$ is

$$A = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

$\mathbf{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$ is a 1-eigenvector of $A$.

The reason for this: $\mathbf{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$ lies in the line $y = mx$, and hence

$$Q_m \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}, \text{ implying that } A \begin{bmatrix} 1 \\ m \end{bmatrix} = 1 \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

## Example (continued)

More generally, any vector $\begin{bmatrix} k \\ km \end{bmatrix}$, $k \neq 0$, lies in the line $y = mx$ and is an eigenvector of $A$.

The perpendicular vector $\begin{bmatrix} -m \\ 1 \end{bmatrix}$ is reflected directly across the line and is thus also an eigenvector for $A$ with eigenvalue $-1$.

## Example (§3.3 Example 7)

Let $\theta$ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of $\theta$, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Find the eigenvalues of $A$.

## Example (§3.3 Example 7)

Let $\theta$ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of $\theta$, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Find the eigenvalues of $A$.

$A$ has no real eigenvectors unless $\theta$ is an integer multiple of $\pi$ $(\pm\pi, \pm 2\pi, \pm 3\pi, \ldots)$ since for other values of $\theta$ there are no invariant lines.

# Summary

1. Eigenvalues and Eigenvectors
2. Geometric Interpretation