Linear Methods (Math 211)
Lecture 24 - §3.2 & 3.3

(with slides adapted from K. Seyffarth)

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Recall

1. Determinants - and Transpose
2. Cramer's Rule
3. Polynomial Interpolation

Today

1. Polynomial Interpolation
2. Vandermonde Determinants
3. Diagonalization
4. Eigenvalues and Eigenvectors

Theorem (§3.2 Theorem 6)

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

The polynomial $p(x)$ is called the interpolating polynomial for the data. We will prove that interpolating polynomials exist and are unique in the next few slides.

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$.

$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$:

$$\begin{aligned}
r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} &= y_1 \\
r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} &= y_2 \\
r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} &= y_3 \\
\vdots \quad &\vdots \\
r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} &= y_n
\end{aligned}$$

The coefficient matrix for this system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

The determinant of a matrix of this form is called a Vandermonde determinant.

The Vandermonde Determinant

Theorem (§3.2 Theorem 7)

Let $x_1, x_2, \ldots, x_n$ *be real numbers,* $n \geq 2$*. The the corresponding Vandermonde determinant is*

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \Pi_{1 \leq j < i \leq n}(x_i - x_j).$$

Example

In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

$$x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$$

giving us the Vandermonde determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

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According to Theorem 7, this determinant is equal to

$$\begin{aligned} &(a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ &= (1-0)(2-0)(2-1)(3-0)(3-1)(3-2) = 2 \cdot 3 \cdot 2 = 12. \end{aligned}$$

As a consequence of Theorem 7, the Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

This means that given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with distinct x_i, then there is a unique interpolating polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}.$$

Example

Let $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$. Find A^{100} efficiently.

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Consider the matrix $P = \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix}$. Observe that P is invertible and that

$$P^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}.$$

Furthermore,

$$P^{-1}AP = \frac{1}{3}\begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix} = D,$$

where D is a diagonal matrix.

Example (continued)

This is significant, because

$$\begin{aligned} P^{-1}AP &= D \\ P(P^{-1}AP)P^{-1} &= PDP^{-1} \\ (PP^{-1})A(PP^{-1}) &= PDP^{-1} \\ IAI &= PDP^{-1} \\ A &= PDP^{-1}, \end{aligned}$$

and

$$\begin{aligned} A^{100} &= (PDP^{-1})^{100} \\ &= (PDP^{-1})(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1}) \\ &= PD(P^{-1}P)D(P^{-1}P)D(P^{-1}\cdots P)DP^{-1} \\ &= PDIDIDI\cdots IDP^{-1} \\ &= PD^{100}P^{-1}. \end{aligned}$$

Example (continued)

Now,

$$D^{100} = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}^{100} = \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix}.$$

Therefore,

$$\begin{aligned} A^{100} &= PD^{100}P^{-1} \\ &= \begin{bmatrix} 1 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2^{100} & 0 \\ 0 & 5^{100} \end{bmatrix} \left(\frac{1}{3}\right) \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2 \cdot 2^{100} + 5^{100} \end{bmatrix} \\ &= \frac{1}{3} \begin{bmatrix} 2^{100} + 2 \cdot 5^{100} & 2^{100} - 2 \cdot 5^{100} \\ 2^{100} - 5^{100} & 2^{101} + 5^{100} \end{bmatrix} \end{aligned}$$

Theorem (§3.3 Theorem 1)

If $A = PDP^{-1}$, *then* $A^k = PD^kP^{-1}$ *for each* $k = 1, 2, 3, \ldots$

The process of finding an invertible matrix P and a diagonal matrix D so that $A = PDP^{-1}$ is referred to as diagonalizing the matrix A, and P is called the diagonalizing matrix for A.

Problem

- *When is it possible to diagonalize a matrix?*
- *How do we find a diagonalizing matrix?*

Eigenvalues and Eigenvectors

Definition

Let A be an $n \times n$ matrix, λ a real number, and $\mathbf{x} \neq 0$ an n-vector. If $A\mathbf{x} = \lambda\mathbf{x}$, then λ is an eigenvalue of A, and $\mathbf{x}$ is an eigenvector of A corresponding to λ, or a λ-eigenvector.

Example

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Then

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3\mathbf{x}.$$

This means that 3 is an eigenvalue of A, and $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of A corresponding to 3 (or a 3-eigenvector of A).

Finding all Eigenvalues and Eigenvectors of a Matrix

Suppose that A is an $n \times n$ matrix, $\mathbf{x} \neq 0$ an n-vector, $\lambda \in \mathbb{R}$, and that $A\mathbf{x} = \lambda\mathbf{x}$. Then

$$\begin{aligned} \lambda\mathbf{x} - A\mathbf{x} &= 0 \\ \lambda I\mathbf{x} - A\mathbf{x} &= 0 \\ (\lambda I - A)\mathbf{x} &= 0 \end{aligned}$$

Since $\mathbf{x} \neq 0$, the matrix $\lambda I - A$ has no inverse, and thus

$$\det(\lambda I - A) = 0.$$

Definition

The characteristic polynomial of an $n \times n$ matrix A is

$$c_A(x) = \det(xI - A).$$

Example

Find the characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

Definition

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Example

Find the characteristic polynomial of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

$$\begin{aligned} c_A(x) &= \det\left(\begin{bmatrix} x & 0 \\ 0 & x \end{bmatrix} - \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}\right) \\ &= \det \begin{bmatrix} x-4 & 2 \\ 1 & x-3 \end{bmatrix} \\ &= (x-4)(x-3) - 2 \\ &= x^2 - 7x + 10 \end{aligned}$$

Theorem (§3.3 Theorem 2)

Let A be an $n \times n$ matrix.

1. *The eigenvalues of A are the roots of $c_A(x)$.*
2. *The λ-eigenvectors X are the nontrivial solutions to $(\lambda I - A)X = 0$.*

Example (continued)

Find the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

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Example (continued)

Find the eigenvalues of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

We have

$$c_A(x) = x^2 - 7x + 10 = (x-2)(x-5),$$

so A has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 5$.

Eigenvectors

Example (continued)

Find the eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

Eigenvectors

Example (continued)

Find the eigenvectors of $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

To find the 2-eigenvectors of A, solve $(2I - A)X = 0$:

$$\left[\begin{array}{rr|r} -2 & 2 & 0 \\ 1 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ -2 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rr|r} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution, in parametric form, is

$$X = \begin{bmatrix} t \\ t \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{ where } t \in \mathbb{R}.$$

Example (continued)

Recall that $A = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$.

To find the 5-eigenvectors of A, solve $(5I - A)X = 0$:

$$\left[\begin{array}{cc|c} 1 & 2 & 0 \\ 1 & 2 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

The general solution, in parametric form, is

$$X = \begin{bmatrix} -2s \\ s \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ where } s \in \mathbb{R}.$$

Summary

1. Polynomial Interpolation
2. Vandermonde Determinants
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