Linear Methods (Math 211)
Lecture 23 - §3.2

(with slides adapted from K. Seyffarth)

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Recall

1. Determinants - Products, inverses and transpose
2. Adjugates

Today

1. Determinants and Transpose
2. Cramer's Rule
3. Polynomial interpolation
4. Vandermonde Determinants

Example (§3.2 Exercise 17)

Let A and B be $n \times n$ matrices. Show that $\det(A + B^T) = \det(A^T + B)$.

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Notice that

$$(A + B^T)^T = A^T + (B^T)^T = A^T + B.$$

Since a matrix and it's transpose have the same determinant

$$\begin{aligned} \det(A + B^T) &= \det((A + B^T)^T) \\ &= \det(A^T + B). \end{aligned}$$

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

- If $\operatorname{adj} A$ exists, then A is invertible.
- If A and B are $n \times n$ matrices, then $\det(AB) = \det(B^T A)$.
- Prove or give a counterexample to the following statement: if $\det A = 1$, then $\operatorname{adj} A = A$.

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

- If $\operatorname{adj} A$ exists, then A is invertible.
  False. $\operatorname{adj} 0$ exists, but 0 is not invertible.
- If A and B are $n \times n$ matrices, then $\det(AB) = \det(B^T A)$.
  True. $\det AB = \det A \cdot \det B = \det B^T \cdot \det A = \det B^T A$.
- Prove or give a counterexample to the following statement: if $\det A = 1$, then $\operatorname{adj} A = A$.
  False. Note that if $\det A = 1$ then $\operatorname{adj} A = A^{-1}$. There are plenty of matrices of determinant 1 for which $A \neq A^{-1}$: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ for example.

Cramer's Rule

If A is an $n \times n$ invertible matrix, then the solution to $A\mathbf{x} = \mathbf{b}$ can be given in terms of determinants of matrices.

Theorem (§3.2 Theorem 5)

Let A be an $n \times n$ invertible matrix, and consider the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$.
We define A_i to be the matrix obtained from A by replacing column i with $\mathbf{b}$. Then setting

$$x_i = \frac{\det A_i}{\det A}$$

gives a solution to $A\mathbf{x} = \mathbf{b}$.

Note: This is a *very* inefficient method for large systems.

Example

Solve for x_3:

$$\begin{aligned} 3x_1 &+ x_2 - x_3 = -1 \\ 5x_1 &+ 2x_2 = 2 \\ x_1 &+ x_2 - x_3 = 1 \end{aligned}$$

Example

Solve for x_3:

$$\begin{array}{rcrcrl} 3x_1 & + & x_2 & - & x_3 & = -1 \\ 5x_1 & + & 2x_2 & & & = 2 \\ x_1 & + & x_2 & - & x_3 & = 1 \end{array}$$

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}.$$

Computing the determinants of these two matrices,

$$\det A = -4 \text{ and } \det A_3 = -6.$$

Therefore, $x_3 = \frac{-6}{-4} = \frac{3}{2}$.

Example (continued)

We can also compute $\det A_1$ and $\det A_2$, where

$$A_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix},$$

and then solve for x_1 and x_2.

We get $x_1 = -1$, $x_2 = \frac{7}{2}$.

Polynomial Interpolation

Example

Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

Polynomial Interpolation

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Given data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, find an interpolating polynomial $p(x)$ of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

We want to find the coefficients r_0, r_1, r_2 and r_3 of

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

so that $p(0) = 1$, $p(1) = 2$, $p(2) = 5$, and $p(3) = 10$.

$$\begin{aligned}
p(0) &= r_0 = 1 \\
p(1) &= r_0 + r_1 + r_2 + r_3 = 2 \\
p(2) &= r_0 + 2r_1 + 4r_2 + 8r_3 = 5 \\
p(3) &= r_0 + 3r_1 + 9r_2 + 27r_3 = 10
\end{aligned}$$

Example (continued)

Solve this system of four equations in the four variables r_0, r_1, r_2 and r_3.

$$\left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & 10 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrrr|r} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x) = 1 + x^2.$$

The estimate is

$$y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \frac{13}{4}.$$

Theorem (§3.2 Theorem 6)

Given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for $i = 1, 2, \ldots, n$.

The polynomial $p(x)$ is called the interpolating polynomial for the data. We will prove that interpolating polynomials exist and are unique in the next few slides.

To find $p(x)$, set up a system of n linear equations in the n variables $r_0, r_1, r_2, \ldots, r_{n-1}$.

$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$:

$$\begin{aligned} r_0 + r_1 x_1 + r_2 x_1^2 + \cdots + r_{n-1} x_1^{n-1} &= y_1 \\ r_0 + r_1 x_2 + r_2 x_2^2 + \cdots + r_{n-1} x_2^{n-1} &= y_2 \\ r_0 + r_1 x_3 + r_2 x_3^2 + \cdots + r_{n-1} x_3^{n-1} &= y_3 \\ \vdots &\ \ \vdots \\ r_0 + r_1 x_n + r_2 x_n^2 + \cdots + r_{n-1} x_n^{n-1} &= y_n \end{aligned}$$

The coefficient matrix for this system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

The determinant of a matrix of this form is called a Vandermonde determinant.

The Vandermonde Determinant

Theorem (§3.2 Theorem 7)

Let $x_1, x_2, \ldots, x_n$ *be real numbers,* $n \geq 2$. *The the corresponding Vandermonde determinant is*

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \Pi_{1 \leq j < i \leq n}(x_i - x_j).$$

Example

In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

$$x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$$

giving us the Vandermonde determinant

$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

Example

In our earlier example with the data points $(0, 1)$, $(1, 2)$, $(2, 5)$ and $(3, 10)$, we have

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$$\begin{vmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & 8 \\ 1 & 3 & 9 & 27 \end{vmatrix}$$

According to Theorem 7, this determinant is equal to

$$\begin{aligned} &(a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3) \\ &= (1 - 0)(2 - 0)(2 - 1)(3 - 0)(3 - 1)(3 - 2) = 2 \times 3 \times 2 \\ &= 12. \end{aligned}$$

As a consequence of Theorem 7, the Vandermonde determinant is nonzero if $a_1, a_2, \ldots, a_n$ are distinct.

This means that given n data points $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ with distinct x_i, then there is a unique interpolating polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}.$$

Summary

1. Determinants and Transpose
2. Cramer's Rule
3. Polynomial interpolation
4. Vandermonde Determinants