Determinants and Transpose	Cramer's Rule	Polynomial interpolation	Vandermonde Determinants

Linear Methods (Math 211) Lecture 23 - §3.2

(with slides adapted from K. Seyffarth)

David Roe

November 4, 2013

Cramer's Rule

Polynomial interpolation 0000

Vandermonde Determinants

Recall

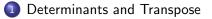
- Oeterminants Products, inverses and transpose
- 2 Adjugates

Cramer's Rule

Polynomial interpolation

Vandermonde Determinants

Today



2 Cramer's Rule





4 Vandermonde Determinants

Determinants and	Transpose
•0	

Cramer's Rule

Polynomial interpolation

Vandermonde Determinants

Example (§3.2 Exercise 17)

Let A and B be $n \times n$ matrices. Show that $det(A + B^T) = det(A^T + B)$.

Determinants and	Transpose
•0	

Cramer's Rule

Polynomial interpolation 0000

Vandermonde Determinants

Example (§3.2 Exercise 17)

Let A and B be $n \times n$ matrices. Show that $det(A + B^T) = det(A^T + B)$.

Notice that

$$(A + B^{T})^{T} = A^{T} + (B^{T})^{T} = A^{T} + B.$$

Since a matrix and it's transpose have the same determinant

$$det(A + B^{T}) = det((A + B^{T})^{T})$$
$$= det(A^{T} + B).$$

Determinants and Transpose	Cramer's Rule	Polynomial interpolation	Vandermonde Determinants
0	000	0000	000

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

- If adj A exists, then A is invertible.
- If A and B are $n \times n$ matrices, then $det(AB) = det(B^T A)$.
- Prove or give a counterexample to the following statement: if det A = 1, then adj A = A.

Determinants and Transpose	Cramer's Rule	Polynomial interpolation	Vandermonde Determinants
0	000	0000	000

Example

For each of the following statements, determine if it is true or false, and supply a proof or a counterexample.

- If adj A exists, then A is invertible.
 False. adj 0 exists, but 0 is not invertible.
- If A and B are n × n matrices, then det(AB) = det(B^TA).
 True. det AB = det A · det B = det B^T · det A = det B^TA.



If A is an $n \times n$ invertible matrix, then the solution to $A\mathbf{x} = \mathbf{b}$ can be given in terms of determinants of matrices.

Theorem ($\S3.2$ Theorem 5)

Let A be an $n \times n$ invertible matrix, and consider the system $A\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}^T$. We define A_i to be the matrix obtained from A by replacing column *i* with **b**. Then setting

$$x_i = \frac{\det A_i}{\det A}$$

gives a solution to $A\mathbf{x} = \mathbf{b}$.

Note: This is a very inefficient method for large systems.

Cramer's Rule ○●○ Polynomial interpolation 0000

Vandermonde Determinants

Example

Solve for x_3 :

$3x_1$	+	<i>x</i> ₂	—	<i>x</i> 3	= -1
$5x_1$	+	$2x_2$			= 2
<i>X</i> 1	+	X2	_	X3	= 1

Cramer's Rule

Polynomial interpolation 0000

Vandermonde Determinants

Example

Solve for x_3 :

By Cramer's rule, $x_3 = \frac{\det A_3}{\det A}$, where

$$A = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } A_3 = \begin{bmatrix} 3 & 1 & -1 \\ 5 & 2 & 2 \\ 1 & 1 & 1 \end{bmatrix}$$

Computing the determinants of these two matrices,

det A = -4 and det $A_3 = -6$.

Therefore, $x_3 = \frac{-6}{-4} = \frac{3}{2}$.

Determinants and Transpose	Cramer's Rule	Polynomial interpolation	Vandermonde Determinants

Example (continued)

We can also compute det A_1 and det A_2 , where

$$A_1 = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix} \text{ and } A_2 = \begin{bmatrix} 3 & -1 & -1 \\ 5 & 2 & 0 \\ 1 & 1 & -1 \end{bmatrix}$$

and then solve for x_1 and x_2 .

We get $x_1 = -1$, $x_2 = \frac{7}{2}$.

Cramer's Rule

Polynomial interpolation •000 Vandermonde Determinants

Polynomial Interpolation

Example

Given data points (0, 1), (1, 2), (2, 5) and (3, 10), find an interpolating polynomial p(x) of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

Cramer's Rule

Polynomial interpolation • 000 Vandermonde Determinants

Polynomial Interpolation

Example

Given data points (0, 1), (1, 2), (2, 5) and (3, 10), find an interpolating polynomial p(x) of degree at most three, and then estimate the value of y corresponding to $x = \frac{3}{2}$.

We want to find the coefficients r_0 , r_1 , r_2 and r_3 of

$$p(x) = r_0 + r_1 x + r_2 x^2 + r_3 x^3$$

so that p(0) = 1, p(1) = 2, p(2) = 5, and p(3) = 10.

$$p(0) = r_0 = 1$$

$$p(1) = r_0 + r_1 + r_2 + r_3 = 2$$

$$p(2) = r_0 + 2r_1 + 4r_2 + 8r_3 = 5$$

$$p(3) = r_0 + 3r_1 + 9r_2 + 27r_3 = 10$$

Cramer's Rule

Polynomial interpolation 0000

Vandermonde Determinants

Example (continued)

Solve this system of four equations in the four variables r_0 , r_1 , r_2 and r_3 .

$$\begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 1 & 1 & 1 & 1 & 2 \\ 1 & 2 & 4 & 8 & 5 \\ 1 & 3 & 9 & 27 & | 10 \end{bmatrix} \to \dots \to \begin{bmatrix} 1 & 0 & 0 & 0 & | & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & | & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore $r_0 = 1$, $r_1 = 0$, $r_2 = 1$, $r_3 = 0$, and so

$$p(x)=1+x^2.$$

The estimate is

$$y = p\left(\frac{3}{2}\right) = 1 + \left(\frac{3}{2}\right)^2 = \frac{13}{4}.$$

Determinants and Transpose	Cramer's Rule	Polynomial interpolation	Vandermonde Determinants
00	000	0000	000

Theorem ($\S3.2$ Theorem 6)

Given n data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with the x_i distinct, there is a unique polynomial

$$p(x) = r_0 + r_1 x + r_2 x^2 + \dots + r_{n-1} x^{n-1}$$

such that $p(x_i) = y_i$ for i = 1, 2, ..., n.

The polynomial p(x) is called the interpolating polynomial for the data. We will prove that interpolating polynomials exist and are unique in the next few slides.

To find p(x), set up a system of *n* linear equations in the *n* variables $r_0, r_1, r_2, \ldots, r_{n-1}$.

Determinants and Transpose $\circ\circ$

 $p(x) = r_0 + r_1 x + r_2 x^2 + \cdots + r_{n-1} x^{n-1}$:

$$r_{0} + r_{1}x_{1} + r_{2}x_{1}^{2} + \dots + r_{n-1}x_{1}^{n-1} = y_{1}$$

$$r_{0} + r_{1}x_{2} + r_{2}x_{2}^{2} + \dots + r_{n-1}x_{2}^{n-1} = y_{2}$$

$$r_{0} + r_{1}x_{3} + r_{2}x_{3}^{2} + \dots + r_{n-1}x_{3}^{n-1} = y_{3}$$

$$\vdots$$

$$r_{0} + r_{1}x_{n} + r_{2}x_{n}^{2} + \dots + r_{n-1}x_{n}^{n-1} = y_{n}$$

The coefficient matrix for this system is

$$\begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix}$$

The determinant of a matrix of this form is called a Vandermonde determinant.

Cramer's Rule

Polynomial interpolation

Vandermonde Determinants •00

The Vandermonde Determinant

Theorem ($\S3.2$ Theorem 7)

Let $x_1, x_2, ..., x_n$ be real numbers, $n \ge 2$. The the corresponding Vandermonde determinant is

$$\det \begin{bmatrix} 1 & x_1 & x_1^2 & \cdots & x_1^{n-1} \\ 1 & x_2 & x_2^2 & \cdots & x_2^{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 & \cdots & x_n^{n-1} \end{bmatrix} = \prod_{1 \le j < i \le n} (x_i - x_j).$$

Determinants and Transpose	Cramer's Rule	Polynomial interpolation	Vandermonde Determinants ○●○
Example			

In our earlier example with the data points (0, 1), (1, 2), (2, 5) and (3, 10), we have

$$x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$$

giving us the Vandermonde determinant

1	0	0	0
1	1	1	1
1 1 1 1	2	4	0 1 8 27
1	3	9	27

Determinants and Transpose	Cramer's Rule 000	Polynomial interpolation	Vandermonde Determinants ○●○
Example			

In our earlier example with the data points (0, 1), (1, 2), (2, 5) and (3, 10), we have

$$x_1 = 0, x_2 = 1, x_3 = 2, x_4 = 3$$

giving us the Vandermonde determinant

1	0	0	0
1	1	1	1
1 1 1 1	2	4	0 1 8 27
1	3	9	27

According to Theorem 7, this determinant is equal to

$$(a_2 - a_1)(a_3 - a_1)(a_3 - a_2)(a_4 - a_1)(a_4 - a_2)(a_4 - a_3)$$

= $(1 - 0)(2 - 0)(2 - 1)(3 - 0)(3 - 1)(3 - 2) = 2 \times 3 \times 2$
= 12.

Determinants and Transpose	Cramer's Rule	Polynomial interpolation	Vandermonde Determinants

As a consequence of Theorem 7, the Vandermonde determinant is nonzero if a_1, a_2, \ldots, a_n are distinct.

This means that given *n* data points $(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n)$ with distinct x_i , then there is a unique interpolating polynomial

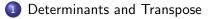
$$p(x) = r_0 + r_1 x + r_2 x^2 + \dots + r_{n-1} x^{n-1}$$

Cramer's Rule

Polynomial interpolation

Vandermonde Determinants

Summary



2 Cramer's Rule





4 Vandermonde Determinants