Linear Methods (Math 211)
Lecture 28 - §3.3 & Midterm Review

(with slides adapted from K. Seyffarth)

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Recall

1. Diagonalization
2. Linear Dynamical Systems
3. Approximate Solutions

Today

Dominant Eigenvalues

Recall that we're studying linear dynamical systems, where $\mathbf{v}_k$ is defined from $\mathbf{v}_{k-1}$ by multiplication by a matrix.

Often, instead of finding an exact formula for $\mathbf{v}_k$, it suffices to estimate $\mathbf{v}_k$ as k gets large.

This can easily be done if A has a dominant eigenvalue with multiplicity one: an eigenvalue λ_1 with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \ldots, n.$$

Suppose that

$$\mathbf{v}_k = PD^kP^{-1}\mathbf{v}_0,$$

and assume that A has a dominant eigenvalue, λ_1, with corresponding basic eigenvector $\mathbf{x}_1$ as the first column of P. For convenience, write $P^{-1}\mathbf{v}_0 = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$.

Then

$$
\begin{aligned}
\mathbf{v}_k &= PD^kP^{-1}\mathbf{v}_0 \\
&= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \\
&= \lambda_1^k\left(b_1\mathbf{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k\mathbf{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k\mathbf{x}_n\right)
\end{aligned}
$$

Now, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for $j = 2, 3, \ldots n$, and thus $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

Therefore, for large values of k, $\mathbf{v}_k \approx \lambda_1^k b_1\mathbf{x}_1$.

Example

Consider an owl population of 100 adult females and 40 juvenile females, and assume we wish to study the population growth.

Imagine that we are biologists and have determined that in general:

1. The number of juvenile females hatched in any year is twice the number of adult females in the year before.
2. Half the adult females in any year survive to the next year.
3. One quarter of the juvenile females survive to adulthood.

Will the owls survive?

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Adults (a_k)	$a_{k+1} = \frac{1}{2}a_k + \frac{1}{4}j_k$
Juveniles (j_k)	$j_{k+1} = 2a_k$

Example (continued)

1. Characteristic polynomial:
$(x - \frac{1}{2})x - (2)(\frac{1}{4}) = x^2 - \frac{1}{2}x - \frac{1}{2} = (x-1)(x+\frac{1}{2})$.
2. Dominant eigenvalue $\lambda = 1$ (the owls will survive!):
$$\begin{bmatrix} -\frac{1}{2} & \frac{1}{4} \\ 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}$$
So an eigenvector is $\begin{bmatrix} 1 & 2 \end{bmatrix}^T$.
3. Eigenvalue $\lambda = -\frac{1}{2}$.
$$\begin{bmatrix} 1 & \frac{1}{4} \\ 2 & \frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{4} \\ 0 & 0 \end{bmatrix}$$
So an eigenvector is $\begin{bmatrix} 1 & -4 \end{bmatrix}^T$.

Example (continued)

So $P = \begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix}$ and $P^{-1} = -\frac{1}{6}\begin{bmatrix} -4 & -1 \\ -2 & 1 \end{bmatrix}$.

$$\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = P^{-1}\mathbf{v}_0 = \frac{1}{6}\begin{bmatrix} 4 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 100 \\ 40 \end{bmatrix} = \begin{bmatrix} \frac{220}{3} \\ \frac{80}{3} \end{bmatrix}$$

Therefore the limiting population after k years is approximately

$$(1)^k b_1 \mathbf{x}_1 = \frac{220}{3}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{220}{3} \\ \frac{440}{3} \end{bmatrix} \approx \begin{bmatrix} 73 \\ 147 \end{bmatrix}.$$

- §2.4 Inverses of matrix transformations
- §2.5 Elementary matrices - inverses, finding U with $A = UR$ or $R = UA$.
- §2.6 Linear transformations - linear $\Leftrightarrow$ matrix, finding the matrix of a transformation, composition and matrix multiplication, rotations and reflections.
- §2.9 Markov chains - finding the transition matrix, regular matrices and the steady state.
- A1 Complex numbers - arithmetic, transforming rectangular $\leftrightarrow$ polar, conjugation and absolute value, multiplication, exponentiation and roots, roots of unity, quadratic equations.
- §3.1 Cofactor expansion - cofactors, row and column expansion, determinants and row operations, triangular matrices.
- §3.2 Determinants and inverses - condition for invertibility, determinants of AB, A^{-1}, A^T and kA, orthogonal matrices, adjugates, Cramer's rule.
- §3.3 Eigenvalues - characteristic polynomial, finding eigenvalues and eigenvectors.

Summary

1. Approximate Solutions

2. Review