Linear Methods (Math 211)
Lecture 27 - §3.3

(with slides adapted from K. Seyffarth)

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Recall

1. Geometric Interpretation of Eigenvalues and Eigenvectors
2. Diagonalization

Today

1. Diagonalization
2. Linear Dynamical Systems
3. Approximate Solutions

Example

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

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$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = -3$ of multiplicity one.

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{x} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$, $s, t \in \mathbb{R}$ so basic eigenvectors corresponding to $\lambda_1 = 1$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Example (continued)

Eigenvectors for $\lambda_2 = -3$: solve $(-3I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{rrr|r} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{x} = \begin{bmatrix} -\frac{1}{4}t \\ 0 \\ t \end{bmatrix}$, $t \in \mathbb{R}$ so a basic eigenvector corresponding to $\lambda_2 = -3$ is

$$\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

Example (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Then P is invertible, and

$$P^{-1}AP = \mathrm{diag}(-3, 1, 1) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example

Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

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Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

First,

$$c_A(x) = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = (x-1)^2(x-2),$$

so A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = 2$ (of multiplicity one).

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{rrr|r} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $\mathbf{x} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$, $s \in \mathbb{R}$.

Since $\lambda_1 = 1$ has multiplicity two, but has only one basic eigenvector, A is not diagonalizable.

Definitions

A **linear dynamical system** consists of

- an $n \times n$ matrix A and an n-vector $\mathbf{v}_0$;
- a matrix recursion defining $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots$ by $\mathbf{v}_{k+1} = A\mathbf{v}_k$; i.e.,

$$\begin{aligned}
\mathbf{v}_1 &= A\mathbf{v}_0 \\
\mathbf{v}_2 &= A\mathbf{v}_1 = A(A\mathbf{v}_0) = A^2\mathbf{v}_0 \\
\mathbf{v}_3 &= A\mathbf{v}_2 = A(A^2\mathbf{v}_0) = A^3\mathbf{v}_0 \\
&\vdots \qquad\qquad\qquad\qquad \vdots \\
\mathbf{v}_k &= A^k\mathbf{v}_0.
\end{aligned}$$

Linear dynamical systems are used, for example, to model the evolution of populations over time.

If A is diagonalizable, then

$$P^{-1}AP = D = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n),$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the (not necessarily distinct) eigenvalues of A.

Thus $A = PDP^{-1}$, and $A^k = PD^kP^{-1}$. Therefore,

$$\mathbf{v}_k = A^k\mathbf{v}_0 = PD^kP^{-1}\mathbf{v}_0.$$

Example

Consider the linear dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } \mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find a formula for $\mathbf{v}_k$.

Example

Consider the linear dynamical system $\mathbf{v}_{k+1} = A\mathbf{v}_k$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } \mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Find a formula for $\mathbf{v}_k$.

First, $c_A(x) = x^2 - x - 1$, so A has eigenvalues $\phi = \frac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \frac{1-\sqrt{5}}{2}$, and thus is diagonalizable.

Solve $(A - \phi I)\mathbf{x} = 0$:

$$\left[\begin{array}{cc|c} -\phi & 1 & 0 \\ 1 & 1-\phi & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \bar{\phi} & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has basic solution $\mathbf{x} = \begin{bmatrix} -\bar{\phi} \\ 1 \end{bmatrix}$.

Example (continued)

Solve $(A - \bar{\phi} I)\mathbf{x} = 0$:

$$\left[\begin{array}{cc|c} -\bar{\phi} & 1 & 0 \\ 1 & 1-\bar{\phi} & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \phi & 0 \\ 0 & 0 & 0 \end{array}\right]$$

has basic solution $\bar{\mathbf{x}} = \begin{bmatrix} -\phi \\ 1 \end{bmatrix}$.

Thus, $P = \begin{bmatrix} -\bar{\phi} & -\phi \\ 1 & 1 \end{bmatrix}$ is a diagonalizing matrix for A,

$$P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \phi \\ -1 & -\bar{\phi} \end{bmatrix}, \text{ and } P^{-1}AP = \begin{bmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{bmatrix}.$$

Example (continued)

Therefore,

$$\begin{aligned}
\mathbf{v}_k &= A^k \mathbf{v}_0 \\
&= PD^kP^{-1}\mathbf{v}_0 \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} -\bar{\phi} & -\phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi & 0 \\ 0 & \bar{\phi} \end{bmatrix}^k \begin{bmatrix} 1 & \phi \\ -1 & -\bar{\phi} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} -\bar{\phi} & -\phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^k & 0 \\ 0 & \bar{\phi}^k \end{bmatrix} \begin{bmatrix} \phi \\ -\bar{\phi} \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} -\bar{\phi} & -\phi \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \phi^{k+1} \\ -\bar{\phi}^{k+1} \end{bmatrix} \\
&= \frac{1}{\sqrt{5}} \begin{bmatrix} \phi^k - \bar{\phi}^k \\ \phi^{k+1} - \bar{\phi}^{k+1} \end{bmatrix}
\end{aligned}$$

Dominant Eigenvalues

Often, instead of finding an exact formula for $\mathbf{v}_k$, it suffices to estimate $\mathbf{v}_k$ as k gets large.

This can easily be done if A has a dominant eigenvalue with multiplicity one: an eigenvalue λ_1 with the property that

$$|\lambda_1| > |\lambda_j| \text{ for } j = 2, 3, \ldots, n.$$

Suppose that

$$\mathbf{v}_k = PD^k P^{-1}\mathbf{v}_0,$$

and assume that A has a dominant eigenvalue, λ_1, with corresponding basic eigenvector $\mathbf{x}_1$ as the first column of P. For convenience, write $P^{-1}\mathbf{v}_0 = \begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix}^T$.

Then

$$\begin{aligned}
\mathbf{v}_k &= PD^kP^{-1}\mathbf{v}_0 \\
&= \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix} \begin{bmatrix} \lambda_1^k & 0 & \cdots & 0 \\ 0 & \lambda_2^k & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda_n^k \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \\
&= b_1\lambda_1^k\mathbf{x}_1 + b_2\lambda_2^k\mathbf{x}_2 + \cdots + b_n\lambda_n^k\mathbf{x}_n \\
&= \lambda_1^k \left(b_1\mathbf{x}_1 + b_2\left(\frac{\lambda_2}{\lambda_1}\right)^k \mathbf{x}_2 + \cdots + b_n\left(\frac{\lambda_n}{\lambda_1}\right)^k \mathbf{x}_n \right)
\end{aligned}$$

Now, $\left|\frac{\lambda_j}{\lambda_1}\right| < 1$ for $j = 2, 3, \ldots n$, and thus $\left(\frac{\lambda_j}{\lambda_1}\right)^k \to 0$ as $k \to \infty$.

Therefore, for large values of k, $\mathbf{v}_k \approx \lambda_1^k b_1 \mathbf{x}_1$.

Example

If

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \text{ and } \mathbf{v}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

estimate $\mathbf{v}_k$ for large values of k.

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estimate $\mathbf{v}_k$ for large values of k.

In our previous example, we found that A has eigenvalues $\phi \approx 1.6$ and $\bar{\phi} \approx -0.6$. Since $|\phi| > |\bar{\phi}|$, ϕ is a dominant eigenvalue.

As before $\mathbf{x} = \begin{bmatrix} -\bar{\phi} \\ 1 \end{bmatrix}$ is a basic eigenvector for ϕ, and $\bar{\mathbf{x}} = \begin{bmatrix} -\phi \\ 1 \end{bmatrix}$ is a basic eigenvector for $\bar{\phi}$, giving us

$$P = \begin{bmatrix} -\bar{\phi} & -\phi \\ 1 & 1 \end{bmatrix}, \text{ and } P^{-1} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & \phi \\ -1 & -\bar{\phi} \end{bmatrix}.$$

Example (continued)

$$P^{-1}\mathbf{v}_0 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & \phi \\ -1 & -\bar{\phi} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} \phi \\ -\bar{\phi} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

For large values of k,

$$\mathbf{v}_k \approx \phi^k b_1 \mathbf{x} = \phi^k \frac{\phi}{\sqrt{5}}\begin{bmatrix} -\bar{\phi} \\ 1 \end{bmatrix} = \begin{bmatrix} \phi^k \\ \phi^{k+1} \end{bmatrix}$$

Let's compare this to the formula for $\mathbf{v}_k$ that we obtained earlier:

$$\mathbf{v}_k = \begin{bmatrix} \phi^k - \bar{\phi}^k \\ \phi^{k+1} - \bar{\phi}^{k+1} \end{bmatrix}$$

Summary

1. Diagonalization
2. Linear Dynamical Systems
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