Linear Methods (Math 211)
Lecture 26 - §3.3

(with slides adapted from K. Seyffarth)

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Recall

1. Eigenvalues and Eigenvectors

Today

1. Geometric Interpretation
2. Diagonalization

Geometric Interpretation of Eigenvalues and Eigenvectors

Let A be a 2×2 matrix. Then A can be interpreted as a linear transformation T_A from $\mathbb{R}^2$ to $\mathbb{R}^2$.

Problem

How does T_A affect the eigenvectors of the matrix?

Geometric Interpretation of Eigenvalues and Eigenvectors

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Definition

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$. We denote by $L_{\mathbf{v}}$ the unique line in $\mathbb{R}^2$ that contains $\mathbf{v}$ and the origin.

Lemma (§3.3 Lemma 1)

Let $\mathbf{v}$ be a nonzero vector in $\mathbb{R}^2$. Then $L_{\mathbf{v}}$ is the set of all scalar multiples of $\mathbf{v}$, i.e.

$$L_{\mathbf{v}} = \mathbb{R}\mathbf{v} = \{t\mathbf{v} \mid t \in \mathbb{R}\}.$$

Definition

Let A be a 2×2 matrix and L a line in $\mathbb{R}^2$ through the origin. Then L is said to be *A-invariant* if the vector $A\mathbf{x}$ lies in L whenever $\mathbf{x}$ lies in L:

- $A\mathbf{x}$ is a scalar multiple of $\mathbf{x}$,
- $A\mathbf{x} = \lambda\mathbf{x}$ for some scalar $\lambda \in \mathbb{R}$,
- $\mathbf{x}$ is an eigenvector of A.

Theorem (§3.3 Theorem 3)

Let A be a 2×2 matrix and let $\mathbf{v} \neq 0$ be a vector in $\mathbb{R}^2$. Then $L_{\mathbf{v}}$ is A-invariant if and only if $\mathbf{v}$ is an eigenvector of A.

This theorem provides a geometrical method for finding the eigenvectors of a 2×2 matrix.

Example (§3.3 Example 6)

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $y = mx$. Find the eigenvalues and eigenvectors of Q_m.

Example (§3.3 Example 6)

Let $m \in \mathbb{R}$ and consider the linear transformation $Q_m : \mathbb{R}^2 \to \mathbb{R}^2$ given by reflection in the line $y = mx$. Find the eigenvalues and eigenvectors of Q_m.

The matrix that induces Q_m is

$$A = \frac{1}{1+m^2}\begin{bmatrix} 1-m^2 & 2m \\ 2m & m^2-1 \end{bmatrix}.$$

$\mathbf{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$ is a 1-eigenvector of A.

The reason for this: $\mathbf{x}_1 = \begin{bmatrix} 1 \\ m \end{bmatrix}$ lies in the line $y = mx$, and hence

$$Q_m \begin{bmatrix} 1 \\ m \end{bmatrix} = \begin{bmatrix} 1 \\ m \end{bmatrix}, \text{ implying that } A \begin{bmatrix} 1 \\ m \end{bmatrix} = 1 \begin{bmatrix} 1 \\ m \end{bmatrix}.$$

Example (continued)

More generally, any vector $\begin{bmatrix} k \\ km \end{bmatrix}$, $k \neq 0$, lies in the line $y = mx$ and is an eigenvector of A.

The perpendicular vector $\begin{bmatrix} -m \\ 1 \end{bmatrix}$ is reflected directly across the line and is thus also an eigenvector for A with eigenvalue -1.

Example (§3.3 Example 7)

Let θ be a real number, and $R_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ rotation through an angle of θ, induced by the matrix

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Find the eigenvalues of A.

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Find the eigenvalues of A.

A has no real eigenvectors unless θ is an integer multiple of π $(\pm\pi, \pm 2\pi, \pm 3\pi, \ldots)$ since for other values of θ there are no invariant lines.

Diagonal Matrices

Notation. An $n \times n$ diagonal matrix

$$D = \begin{bmatrix} a_1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_2 & 0 & \cdots & 0 & 0 \\ 0 & 0 & a_3 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{n-1} & 0 \\ 0 & 0 & 0 & \cdots & 0 & a_n \end{bmatrix}$$

is written $\text{diag}(a_1, a_2, a_3, \ldots, a_{n-1}, a_n)$.

Recall that if A is an $n \times n$ matrix and P is an invertible $n \times n$ matrix so that $P^{-1}AP$ is diagonal, then P is called a diagonalizing matrix of A, and A is diagonalizable.

Theorem (§3.3 Theorem 4 & 5 & 6)

Let A be an $n \times n$ matrix.

1. *A is diagonalizable if and only if it has eigenvectors $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_n$ so that*

$$P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix}$$

   is invertible. This occurs precisely when the total number of basic eigenvectors equals n.

2. *If P is invertible, then*

$$P^{-1}AP = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$$

   where λ_i is the eigenvalue of A corresponding to the eigenvector $\mathbf{x}_i$, i.e., $A\mathbf{x}_i = \lambda_i \mathbf{x}_i$.

3. *If all of the eigenvalues of A are distinct then A is diagonalizable.*

Example

$$A = \begin{bmatrix} 3 & -4 & 2 \\ 1 & -2 & 2 \\ 1 & -5 & 5 \end{bmatrix}$$ has eigenvalues and basic eigenvectors

$$\lambda_1 = 3 \text{ and } \mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}; \lambda_2 = 2 \text{ and } \mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}; \lambda_3 = 1 \text{ and } \mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

Diagonalize A.

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Diagonalize A.

Let $P = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 2 & 1 & 1 \end{bmatrix}$. By Theorem 4 and 5,

$$P^{-1}AP = \text{diag}(3, 2, 1) = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Note. It is not always possible to find n eigenvectors so that P is invertible.

Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$. Is A diagonalizable?

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Example

Let $A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 6 & -6 \\ 1 & 2 & -1 \end{bmatrix}$. Is A diagonalizable?

Then

$$c_A(x) = \begin{vmatrix} x-1 & 2 & -3 \\ -2 & x-6 & 6 \\ -1 & -2 & x+1 \end{vmatrix} = \cdots = (x-2)^3.$$

A has only one eigenvalue, $\lambda_1 = 2$, with multiplicity three.

To find the 2-eigenvectors of A, solve the system $(2I - A)\mathbf{x} = 0$.

Example (continued)

$$\left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ -2 & -4 & 6 & 0 \\ -1 & -2 & 3 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 2 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The general solution in parametric form is

$$\mathbf{x} = \begin{bmatrix} -2s+3t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}, s, t \in \mathbb{R}.$$

Since the system has only two basic solutions, there are only two basic eigenvectors, implying that the matrix A is not diagonalizable.

Example

Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

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Diagonalize, if possible, the matrix $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & -3 \end{bmatrix}$.

$$c_A(x) = \det(xI - A) = \begin{vmatrix} x-1 & 0 & -1 \\ 0 & x-1 & 0 \\ 0 & 0 & x+3 \end{vmatrix} = (x-1)^2(x+3).$$

A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = -3$ of multiplicity one.

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{ccc|c} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{x} = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$, $s, t \in \mathbb{R}$ so basic eigenvectors corresponding to $\lambda_1 = 1$ are

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Example (continued)

Eigenvectors for $\lambda_2 = -3$: solve $(-3I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{rrr|r} -4 & 0 & -1 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$\mathbf{x} = \begin{bmatrix} -\frac{1}{4}t \\ 0 \\ t \end{bmatrix}$, $t \in \mathbb{R}$ so a basic eigenvector corresponding to $\lambda_2 = -3$ is

$$\begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$$

Example (continued)

Let

$$P = \begin{bmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \\ 4 & 0 & 0 \end{bmatrix}.$$

Then P is invertible, and

$$P^{-1}AP = \operatorname{diag}(-3, 1, 1) = \begin{bmatrix} -3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Example

Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

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Show that $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ is not diagonalizable.

First,

$$c_A(x) = \begin{vmatrix} x-1 & -1 & 0 \\ 0 & x-1 & 0 \\ 0 & 0 & x-2 \end{vmatrix} = (x-1)^2(x-2),$$

so A has eigenvalues $\lambda_1 = 1$ of multiplicity two; $\lambda_2 = 2$ (of multiplicity one).

Example (continued)

Eigenvectors for $\lambda_1 = 1$: solve $(I - A)\mathbf{x} = 0$.

$$\left[\begin{array}{rrr|r} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, $\mathbf{x} = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}$, $s \in \mathbb{R}$.

Since $\lambda_1 = 1$ has multiplicity two, but has only one basic eigenvector, A is not diagonalizable.

Summary