Recall

1. Elementary Row Operations
2. Triangular Matrices
3. Multiplying by scalars
4. Block Matrices
5. More examples
Today

1. Products, inverses and transpose

2. Adjugates
Theorem (§3.2 Theorem 1 – Product Theorem)

If $A$ and $B$ are $n \times n$ matrices, then

$$\det(AB) = \det A \det B.$$ 

Theorem (§3.2 Theorem 2)

An $n \times n$ matrix $A$ is invertible if and only if $\det A \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det A}.$$
Example

Find all values of $c$ for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

Therefore, $A$ is invertible for all $c \neq 0, 3, -3$. 

\[ \text{det}(A) = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 2 \\ c & 5 \end{vmatrix} = c(10 - c^2) - c(10 - c^2) = c(9 - c^2) = c(3 - c)(3 + c). \]
Example

Find all values of $c$ for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

$$\det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} = c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix}$$

$$= c(10 - c^2) - c = c(9 - c^2) = c(3 - c)(3 + c).$$

Therefore, $A$ is invertible for all $c \neq 0, 3, -3$. 
Theorem (§3.2 Theorem 3)

If $A$ is an $n \times n$ matrix, then $\det(A^T) = \det A$. 
Example

Suppose $A$ is a $3 \times 3$ matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$
Example

Suppose $A$ is a $3 \times 3$ matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

First,

$$\det(2A^{-1}) = -4$$

$$2^3 \det(A^{-1}) = -4$$

$$\frac{1}{\det A} = \frac{-4}{8} = \frac{-1}{2}$$

Therefore, $\det A = -2.$
Example (continued)

Now,

\[
\det(A^3(B^{-1})^T) = -4
\]
\[
(\det A)^3 \det(B^{-1}) = -4
\]
\[
(-2)^3 \det(B^{-1}) = -4
\]
\[
(-8) \det(B^{-1}) = -4
\]
\[
\frac{1}{\det B} = \frac{-4}{-8} = \frac{1}{2}
\]

Therefore, \( \det B = 2 \).
Example

Suppose $A$, $B$ and $C$ are $4 \times 4$ matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$ 

Find $\det(2A^2(B^{-1})(C^T)^3B(A^{-1}))$. 

$$\det(2A^2(B^{-1})(C^T)^3B(A^{-1})) = 16(\det A)(\det C)^3 = 16 \times (-1) \times 1^3 = -16.$$
Example

Suppose \( A, B \) and \( C \) are \( 4 \times 4 \) matrices with

\[
\det A = -1, \quad \det B = 2, \quad \text{and} \quad \det C = 1.
\]

Find \( \det(2A^2(B^{-1})(C^T)^3B(A^{-1})) \).

\[
\det(2A^2(B^{-1})(C^T)^3B(A^{-1})) \\
= 2^4(\det A)^2 \frac{1}{\det B} (\det C)^3(\det B) \frac{1}{\det A} \\
= 16(\det A)(\det C)^3 \\
= 16 \times (-1) \times 1^3 \\
= -16.
\]
A square matrix $A$ is **orthogonal** if and only if $A^T = A^{-1}$. What are the possible values of det $A$ if $A$ is orthogonal?

Since $A^T = A^{-1}$, \( \det(A^T) = \det(A^{-1}) \).

\[
\det(A) = 1
\]

Thus \( \det(A) = \pm 1 \), i.e., \( \det(A) = 1 \) or \( \det(A) = -1 \).
Example (§3.2 Example 5)

A square matrix $A$ is **orthogonal** if and only if $A^T = A^{-1}$. What are the possible values of $\det A$ if $A$ is orthogonal?

Since $A^T = A^{-1}$,

$$
\det A^T = \det(A^{-1})
$$

$$
\det A = \frac{1}{\det A}
$$

$$(\det A)^2 = 1
$$

Thus $\det A = \pm 1$, i.e., $\det A = 1$ or $\det A = -1$. 

Adjugates

For a $2 \times 2$ matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have already seen the adjugate of $A$ defined as

$$\text{adj } A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and observed that

$$A(\text{adj } A) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} = (\det A) I_2$$

Furthermore, if $\det A \neq 0$, then $A$ is invertible and

$$A^{-1} = \frac{1}{\det A} \text{adj } A.$$
Recall the cofactor \( c_{ij}(A) = (-1)^{i+j} \det(A_{ij}). \)

**Definition**

If \( A \) is an \( n \times n \) matrix, then

\[
\text{adj} \ A = \left[ c_{ij}(A) \right]^T,
\]

where \( c_{ij}(A) \) is the \((i, j)\)-cofactor of \( A \), i.e., \( \text{adj} \ A \) is the transpose of the cofactor matrix (matrix of cofactors).
Example

Find $\text{adj } A$ when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Solution.

$\text{adj } A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$

Notice that $A \cdot (\text{adj } A) = \begin{bmatrix} 180 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 180 \end{bmatrix}$. 
Example

Find adj $A$ when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

Solution.

$\text{adj } A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$

Notice that

$A(\text{adj } A) = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$

$= \begin{bmatrix} 180 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 180 \end{bmatrix}$
Example (continued)

Also,

\[
\begin{vmatrix}
2 & 1 & 3 \\
5 & -7 & 1 \\
3 & 0 & -6 \\
\end{vmatrix}
\]

\[
\begin{vmatrix}
2 & 1 & 3 \\
19 & 0 & 22 \\
3 & 0 & -6 \\
\end{vmatrix}
\]

\[
= \begin{vmatrix}
-1 & 19 & 22 \\
3 & -6 \\
\end{vmatrix}
\]

\[
= 180,
\]

so in this example, we see that

\[
A(\text{adj } A) = (\text{det } A)I
\]
The Adjugate Formula

**Theorem (§3.2 Theorem 4)**

*If $A$ is an $n \times n$ matrix, then*

$$A(\text{adj} \ A) = (\text{det} \ A)I = (\text{adj} \ A)A.$$  

*Furthermore, if $\text{det} \ A \neq 0$, then*

$$A^{-1} = \frac{1}{\text{det} \ A} \text{adj} \ A.$$  

**Note.** Except in the case of a $2 \times 2$ matrix, the adjugate formula is a very inefficient method for computing the inverse of a matrix; the matrix inversion algorithm is much more practical. However, the adjugate formula is of theoretical significance.
Example (§3.2 Example 8)

For an $n \times n$ matrix $A$, show that $\det(\text{adj } A) = (\det A)^{n-1}$. 
Example (§3.2 Example 8)

For an \( n \times n \) matrix \( A \), show that \( \det(\text{adj} \ A) = (\det A)^{n-1} \).

Using the adjugate formula,

\[
A(\text{adj} \ A) = (\det A)I
\]
\[
\det(A(\text{adj} \ A)) = \det((\det A)I)
\]
\[
(\det A) \times \det(\text{adj} \ A) = (\det A)^n(\det I)
\]
\[
(\det A) \times \det(\text{adj} \ A) = (\det A)^n
\]

If \( \det A \neq 0 \), then divide both sides of the last equation by \( \det A \):

\[
\det(\text{adj} \ A) = (\det A)^{n-1}.
\]
Example (continued)

In the case that $\det A = 0$, I claim the adjugate is not invertible (and thus $\det(\text{adj } A) = 0$).

Why? If $\text{adj } A$ were invertible then multiplying

$$A(\text{adj } A) = 0$$

by its inverse would give $A = 0$. But then $\text{adj } A = 0$ and thus not invertible.
Summary

1. Products, inverses and transpose

2. Adjugates