Linear Methods (Math 211)
Lecture 22 - §3.2

(with slides adapted from K. Seyffarth)

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Recall

1. Elementary Row Operations
2. Triangular Matrices
3. Multiplying by scalars
4. Block Matrices
5. More examples

Today

Theorem (§3.2 Theorem 1 – Product Theorem)

If A and B are $n \times n$ matrices, then

$$\det(AB) = \det A \det B.$$

Theorem (§3.2 Theorem 2)

An $n \times n$ matrix A is invertible if and only if $\det A \neq 0$. In this case,

$$\det(A^{-1}) = \frac{1}{\det A}.$$

Example

Find all values of c for which $A = \begin{bmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{bmatrix}$ is invertible.

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$$\begin{aligned} \det A = \begin{vmatrix} c & 1 & 0 \\ 0 & 2 & c \\ -1 & c & 5 \end{vmatrix} &= c \begin{vmatrix} 2 & c \\ c & 5 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 0 \\ 2 & c \end{vmatrix} \\ &= c(10 - c^2) - c = c(9 - c^2) = c(3 - c)(3 + c). \end{aligned}$$

Therefore, A is invertible for all $c \neq 0, 3, -3$.

Theorem (§3.2 Theorem 3)

If A *is an* $n \times n$ *matrix, then* $\det(A^T) = \det A$.

Example

Suppose A is a 3×3 matrix. Find $\det A$ and $\det B$ if

$$\det(2A^{-1}) = -4 = \det(A^3(B^{-1})^T).$$

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First,

$$\begin{aligned} \det(2A^{-1}) &= -4 \\ 2^3 \det(A^{-1}) &= -4 \\ \frac{1}{\det A} &= \frac{-4}{8} = -\frac{1}{2} \end{aligned}$$

Therefore, $\det A = -2$.

Example (continued)

Now,

$$\begin{aligned}
\det(A^3(B^{-1})^T) &= -4 \\
(\det A)^3 \det(B^{-1}) &= -4 \\
(-2)^3 \det(B^{-1}) &= -4 \\
(-8) \det(B^{-1}) &= -4 \\
\frac{1}{\det B} &= \frac{-4}{-8} = \frac{1}{2}
\end{aligned}$$

Therefore, $\det B = 2$.

Example

Suppose A, B and C are 4×4 matrices with

$$\det A = -1, \det B = 2, \text{ and } \det C = 1.$$

Find $\det(2A^2(B^{-1})(C^T)^3B(A^{-1}))$.

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$$\begin{aligned}
&\det(2A^2(B^{-1})(C^T)^3B(A^{-1})) \\
&\qquad = 2^4(\det A)^2\frac{1}{\det B}(\det C)^3(\det B)\frac{1}{\det A} \\
&\qquad = 16(\det A)(\det C)^3 \\
&\qquad = 16 \times (-1) \times 1^3 \\
&\qquad = -16.
\end{aligned}$$

Example (§3.2 Example 5)

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Since $A^T = A^{-1}$,

$$\begin{aligned} \det A^T &= \det(A^{-1}) \\ \det A &= \frac{1}{\det A} \\ (\det A)^2 &= 1 \end{aligned}$$

Thus $\det A = \pm 1$, i.e., $\det A = 1$ or $\det A = -1$.

Adjugates

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we have already seen the adjugate of A defined as

$$\operatorname{adj} A = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

and observed that

$$\begin{aligned} A(\operatorname{adj} A) &= \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \\ &= \begin{bmatrix} ad - bc & 0 \\ 0 & ad - bc \end{bmatrix} \\ &= (\det A) I_2 \end{aligned}$$

Furthermore, if $\det A \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Adjugates

Recall the cofactor $c_{ij}(A) = (-1)^{i+j}\det(A_{ij})$.

Definition

If A is an $n \times n$ matrix, then

$$\operatorname{adj} A = \left[c_{ij}(A)\right]^T,$$

where $c_{ij}(A)$ is the (i, j)-cofactor of A, i.e., $\operatorname{adj} A$ is the transpose of the cofactor matrix (matrix of cofactors).

Example

Find $\operatorname{adj} A$ when $A = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix}$.

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Solution.

$$\text{adj}\, A = \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$$

Notice that

$$A(\text{adj}\, A) = \begin{bmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{bmatrix} \begin{bmatrix} 42 & 6 & 22 \\ 33 & -21 & 13 \\ 21 & 3 & -19 \end{bmatrix}$$
$$= \begin{bmatrix} 180 & 0 & 0 \\ 0 & 180 & 0 \\ 0 & 0 & 180 \end{bmatrix}$$

Example (continued)

Also,

$$\begin{aligned}\det A &= \begin{vmatrix} 2 & 1 & 3 \\ 5 & -7 & 1 \\ 3 & 0 & -6 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 1 & 3 \\ 19 & 0 & 22 \\ 3 & 0 & -6 \end{vmatrix} \\ &= (-1) \begin{vmatrix} 19 & 22 \\ 3 & -6 \end{vmatrix} \\ &= 180,\end{aligned}$$

so in this example, we see that

$$A(\operatorname{adj} A) = (\det A)I$$

The Adjugate Formula

Theorem (§3.2 Theorem 4)

If A is an $n \times n$ matrix, then

$$A(\operatorname{adj} A) = (\det A)I = (\operatorname{adj} A)A.$$

Furthermore, if $\det A \neq 0$, then

$$A^{-1} = \frac{1}{\det A} \operatorname{adj} A.$$

Note. Except in the case of a 2×2 matrix, the adjugate formula is a very inefficient method for computing the inverse of a matrix; the matrix inversion algorithm is much more practical. However, the adjugate formula is of theoretical significance.

Example (§3.2 Example 8)

For an $n \times n$ matrix A, show that $\det(\operatorname{adj} A) = (\det A)^{n-1}$.

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Using the adjugate formula,

$$\begin{aligned} A(\operatorname{adj} A) &= (\det A)I \\ \det(A(\operatorname{adj} A)) &= \det((\det A)I) \\ (\det A) \times \det(\operatorname{adj} A) &= (\det A)^n(\det I) \\ (\det A) \times \det(\operatorname{adj} A) &= (\det A)^n \end{aligned}$$

If $\det A \neq 0$, then divide both sides of the last equation by $\det A$:

$$\det(\operatorname{adj} A) = (\det A)^{n-1}.$$

Example (continued)

In the case that $\det A = 0$, I claim the adjugate is not invertible (and thus $\det(\operatorname{adj} A) = 0$).

Why? If $\operatorname{adj} A$ were invertible then multiplying

$$A(\operatorname{adj} A) = 0$$

by its inverse would give $A = 0$. But then $\operatorname{adj} A = 0$ and thus not invertible.

Summary

1. Products, inverses and transpose
2. Adjugates