MATH 211 – Fall 2012

Lecture Notes

K. Seyffarth

Section 4.3

§4.3 – More on the Cross Product

Let $\vec{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$.

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$$\begin{aligned}
\vec{u} \bullet (\vec{v} \times \vec{w}) &= \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} \bullet \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix} \\
&= x_0(y_1 z_2 - z_1 y_2) - y_0(x_1 z_2 - z_1 x_2) + z_0(x_1 y_2 - y_1 x_2) \\
&= x_0 \begin{vmatrix} y_1 & y_2 \\ z_1 & z_2 \end{vmatrix} - y_0 \begin{vmatrix} x_1 & x_2 \\ z_1 & z_2 \end{vmatrix} + z_0 \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} \\
&= \begin{vmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{vmatrix}.
\end{aligned}$$

Theorem (§4.3 Theorem 1)

If $\vec{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. *Then*

$$\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \begin{bmatrix} x_0 & x_1 & x_2 \\ y_0 & y_1 & y_2 \\ z_0 & z_1 & z_2 \end{bmatrix}.$$

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If $\vec{u} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$, *and* $\vec{w} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$. *Then*

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Shorthand: $\vec{u} \bullet (\vec{v} \times \vec{w}) = \det \begin{bmatrix} \vec{u} & \vec{v} & \vec{w} \end{bmatrix}$.

Properties of the Cross Product

Theorem (§4.3 Theorem 2)

Let $\vec{u}, \vec{v}$ and $\vec{w}$ be in $\mathbb{R}^3$.

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6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ *for any scalar* k*.*

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6. $(k\vec{u}) \times \vec{v} = k(\vec{u} \times \vec{v}) = \vec{u} \times (k\vec{v})$ *for any scalar* k.
7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

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7. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$*.*
8. $(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$*.*

The Lagrange Identity

Theorem (§4.3 Theorem 3)

If $\vec{u}, \vec{v} \in \mathbb{R}^3$, *then*

$$||\vec{u} \times \vec{v}||^2 = ||\vec{u}||^2 ||\vec{v}||^2 - (\vec{u} \bullet \vec{v})^2.$$

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Proof.

Write $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$, and work out all the terms. □

As a consequence of the Lagrange Identity and the fact that

$$\vec{u} \bullet \vec{v} = ||\vec{u}|| \; ||\vec{v}|| \cos\theta,$$

we have

$$\begin{aligned} ||\vec{u} \times \vec{v}||^2 &= ||\vec{u}||^2||\vec{v}||^2 - (\vec{u} \bullet \vec{v})^2 \\ &= ||\vec{u}||^2||\vec{v}||^2 - ||\vec{u}||^2||\vec{v}||^2 \cos^2\theta \\ &= ||\vec{u}||^2||\vec{v}||^2(1 - \cos^2\theta) \\ &= ||\vec{u}||^2||\vec{v}||^2 \sin^2\theta. \end{aligned}$$

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Taking square roots on both sides yields,

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Note that since $0 \leq \theta \leq \pi$, $\sin\theta \geq 0$.

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If $\theta = 0$ or $\theta = \pi$, then $\sin\theta = 0$, and $||\vec{u} \times \vec{v}|| = 0$. This is consistent with our earlier observation that if $\vec{u}$ and $\vec{v}$ are parallel, then $\vec{u} \times \vec{v} = \vec{0}$.

Theorem (§4.3 Theorem 4)

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$, and let θ denote the angle between $\vec{u}$ and $\vec{v}$.

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Let $\vec{u}$ and $\vec{v}$ be nonzero vectors in $\mathbb{R}^3$, and let θ denote the angle between $\vec{u}$ and $\vec{v}$.

1. $||\vec{u} \times \vec{v}|| = ||\vec{u}|| \; ||\vec{v}|| \sin\theta$, *and is the area of the parallelogram defined by $\vec{u}$ and $\vec{v}$.*

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2. *$\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.*

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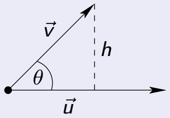
Proof of area of parallelogram.

The area of the parallelogram defined by $\vec{u}$ and $\vec{v}$ is $||\vec{u}||h$, where h is the height of the parallelogram.



□

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1. $||\vec{u} \times \vec{v}|| = ||\vec{u}||\ ||\vec{v}|| \sin\theta$*, and is the area of the parallelogram defined by* $\vec{u}$ *and* $\vec{v}$*.*
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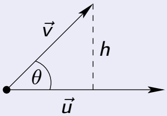

$\sin\theta = \frac{h}{||\vec{v}||}$, implying that $h = ||\vec{v}|| \sin\theta$.

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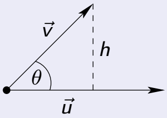

$\sin\theta = \frac{h}{||\vec{v}||}$, implying that $h = ||\vec{v}|| \sin\theta$. Therefore, the area is

$$||\vec{u}||\ ||\vec{v}|| \sin\theta.$$

□

Theorem (§4.3 Theorem 5)

The volume of the parallelepiped determined by the three vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ in $\mathbb{R}^3$ is

$$|\vec{w} \bullet (\vec{u} \times \vec{v})|.$$

Example (§4.3 Exercise 4(a))

Find the area of the triangle having vertices $A(3, -1, 2)$, $B(1, 1, 0)$ and $C(1, 2, -1)$.

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Find the area of the triangle having vertices $A(3, -1, 2)$, $B(1, 1, 0)$ and $C(1, 2, -1)$.

Solution. The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

Example (§4.3 Exercise 4(a))

Find the area of the triangle having vertices $A(3,-1,2)$, $B(1,1,0)$ and $C(1,2,-1)$.

Solution. The area of the triangle is half the area of the parallelogram defined by $\overrightarrow{AB}$ and $\overrightarrow{AC}$.

$$\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} \text{ and } \overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}.$$

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Find the area of the triangle having vertices $A(3, -1, 2)$, $B(1, 1, 0)$ and $C(1, 2, -1)$.

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$\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$. Therefore

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

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$\overrightarrow{AB} = \begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix}$ and $\overrightarrow{AC} = \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}$. Therefore

$$\overrightarrow{AB} \times \overrightarrow{AC} = \begin{bmatrix} 0 \\ -2 \\ -2 \end{bmatrix},$$

so the area of the triangle is $\frac{1}{2}||\overrightarrow{AB} \times \overrightarrow{AC}|| = \sqrt{2}$.

Example (§4.3 Exercise 5(a))

Find the volume of the parallelepiped determined by the vectors

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \ \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \text{ and } \vec{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}.$$

Example (§4.3 Exercise 5(a))

Find the volume of the parallelepiped determined by the vectors $\vec{u} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, and $\vec{w} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$.

Solution. The volume of the parallelepiped is

$$|\vec{u} \bullet (\vec{v} \times \vec{w})| = \left| \det \begin{bmatrix} 2 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 2 & -1 \end{bmatrix} \right| = 2.$$