UNIVERSITY OF CALGARY

MATH 211 – Fall 2012

Lecture Notes

K. Seyffarth

Section 4.2

§4.2 – Projections and Planes

Definition

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \bullet \vec{v} = x_1x_2 + y_1y_2 + z_1z_2,$$

i.e., $\vec{u} \bullet \vec{v}$ is a scalar.

Definition

Let $\vec{u} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}$ be vectors in $\mathbb{R}^3$. The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \bullet \vec{v} = x_1x_2 + y_1y_2 + z_1z_2,$$

i.e., $\vec{u} \bullet \vec{v}$ is a scalar.

Note. Another way to think about the dot product is as the 1×1 matrix

$$\vec{u}^T\vec{v} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = \begin{bmatrix} x_1x_2 + y_1y_2 + z_1z_2 \end{bmatrix}.$$

Properties of the Dot Product

Theorem (§4.2 Theorem 1)

Let $\vec{u}, \vec{v}, \vec{w}$ *be vectors in* $\mathbb{R}^3$ *(or* $\mathbb{R}^2$*) and let* $k \in \mathbb{R}$.

Properties of the Dot Product

Theorem (§4.2 Theorem 1)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \bullet \vec{v}$ *is a real number.*

Properties of the Dot Product

Theorem (§4.2 Theorem 1)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \bullet \vec{v}$ *is a real number.*
2. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$. *(commutative property)*

Properties of the Dot Product

Theorem (§4.2 Theorem 1)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. *$\vec{u} \bullet \vec{v}$ is a real number.*
2. *$\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$.* *(commutative property)*
3. *$\vec{u} \bullet \vec{0} = 0$.*

Properties of the Dot Product

Theorem (§4.2 Theorem 1)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \bullet \vec{v}$ *is a real number.*
2. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$. *(commutative property)*
3. $\vec{u} \bullet \vec{0} = 0$.
4. $\vec{u} \bullet \vec{u} = ||\vec{u}||^2$.

Properties of the Dot Product

Theorem (§4.2 Theorem 1)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

1. $\vec{u} \bullet \vec{v}$ *is a real number.*
2. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$. *(commutative property)*
3. $\vec{u} \bullet \vec{0} = 0$.
4. $\vec{u} \bullet \vec{u} = ||\vec{u}||^2$.
5. $(k\vec{u}) \bullet \vec{v} = k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v})$. *(associative property)*

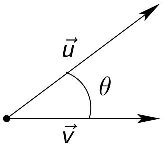
Properties of the Dot Product

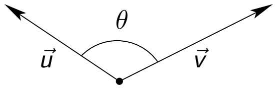
Theorem (§4.2 Theorem 1)

Let $\vec{u}, \vec{v}, \vec{w}$ be vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$) and let $k \in \mathbb{R}$.

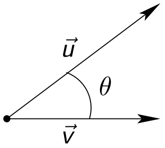
1. $\vec{u} \bullet \vec{v}$ *is a real number.*
2. $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$. *(commutative property)*
3. $\vec{u} \bullet \vec{0} = 0$.
4. $\vec{u} \bullet \vec{u} = ||\vec{u}||^2$.
5. $(k\vec{u}) \bullet \vec{v} = k(\vec{u} \bullet \vec{v}) = \vec{u} \bullet (k\vec{v})$. *(associative property)*
6. $\vec{u} \bullet (\vec{v} + \vec{w}) = \vec{u} \bullet \vec{v} + \vec{u} \bullet \vec{w}$. *(distributive properties)*
   $\vec{u} \bullet (\vec{v} - \vec{w}) = \vec{u} \bullet \vec{v} - \vec{u} \bullet \vec{w}$.

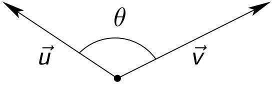
Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$), positioned so they have the same tail. Then there is a unique angle θ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$.





Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$), positioned so they have the same tail. Then there is a unique angle θ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$.



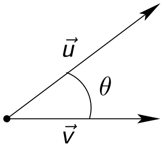


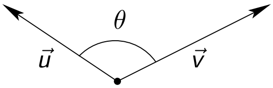
Theorem (§4.2 Theorem 2)

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$), positioned so they have the same tail. Then there is a unique angle θ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$.





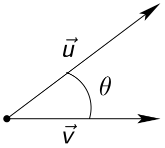
Theorem (§4.2 Theorem 2)

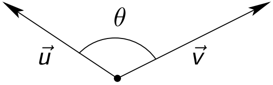
Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

- This is an intrinsic description of the dot product.

Let $\vec{u}$ and $\vec{v}$ be two vectors in $\mathbb{R}^3$ (or $\mathbb{R}^2$), positioned so they have the same tail. Then there is a unique angle θ between $\vec{u}$ and $\vec{v}$ with $0 \leq \theta \leq \pi$.





Theorem (§4.2 Theorem 2)

Let $\vec{u}$ and $\vec{v}$ be nonzero vectors, and let θ denote the angle between $\vec{u}$ and $\vec{v}$. Then

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

- This is an intrinsic description of the dot product.
- The proof uses the Law of Cosines, which is a generalization of the **Pythagorean Theorem**.

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

- If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.

$$\vec{u} \bullet \vec{v} = ||\vec{u}|| \; ||\vec{v}|| \cos\theta.$$

- If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$.

$$\vec{u} \bullet \vec{v} = ||\vec{u}|| \; ||\vec{v}|| \cos\theta.$$

- If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$.
- If $\frac{\pi}{2} < \theta \leq \pi$, then $\cos\theta < 0$.

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

- If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$.
- If $\frac{\pi}{2} < \theta \leq \pi$, then $\cos\theta < 0$.

Therefore, for nonzero vectors $\vec{u}$ and $\vec{v}$,

- $\vec{u} \bullet \vec{v} > 0$ if and only if $0 \leq \theta < \frac{\pi}{2}$.

$$\vec{u} \bullet \vec{v} = ||\vec{u}|| \; ||\vec{v}|| \cos\theta.$$

- If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$.
- If $\frac{\pi}{2} < \theta \leq \pi$, then $\cos\theta < 0$.

Therefore, for nonzero vectors $\vec{u}$ and $\vec{v}$,

- $\vec{u} \bullet \vec{v} > 0$ if and only if $0 \leq \theta \ < \frac{\pi}{2}$.
- $\vec{u} \bullet \vec{v} = 0$ if and only if $\theta = \frac{\pi}{2}$.

$$\vec{u} \bullet \vec{v} = ||\vec{u}||\ ||\vec{v}|| \cos\theta.$$

- If $0 \leq \theta < \frac{\pi}{2}$, then $\cos\theta > 0$.
- If $\theta = \frac{\pi}{2}$, then $\cos\theta = 0$.
- If $\frac{\pi}{2} < \theta \leq \pi$, then $\cos\theta < 0$.

Therefore, for nonzero vectors $\vec{u}$ and $\vec{v}$,

- $\vec{u} \bullet \vec{v} > 0$ if and only if $0 \leq \theta < \frac{\pi}{2}$.
- $\vec{u} \bullet \vec{v} = 0$ if and only if $\theta = \frac{\pi}{2}$.
- $\vec{u} \bullet \vec{v} < 0$ if and only if $\frac{\pi}{2} < \theta \leq \pi$.

Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\theta = \frac{\pi}{2}$.

Definition

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$ or $\theta = \frac{\pi}{2}$.

Theorem (§4.2 Theorem 3)

Vectors $\vec{u}$ and $\vec{v}$ are orthogonal if and only if $\vec{u} \bullet \vec{v} = 0$.

Example

Find the angle between $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Example

Find the angle between $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Solution.
$\vec{u} \bullet \vec{v} = 1$, $||\vec{u}|| = \sqrt{2}$ and $||\vec{v}|| = \sqrt{2}$.
Therefore, by **Theorem 2**,

$$\cos\theta = \frac{\vec{u} \bullet \vec{v}}{||\vec{u}|| \; ||\vec{v}||} = \frac{1}{\sqrt{2}\sqrt{2}} = \frac{1}{2}.$$

Since $0 \leq \theta \leq \pi$, $\theta = \frac{\pi}{3}$.

Therefore, the angle between $\vec{u}$ and $\vec{v}$ is $\frac{\pi}{3}$.

Example

Find the angle between $\vec{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

Example

Find the angle between $\vec{u} = \begin{bmatrix} 7 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$.

Solution.

$\vec{u} \bullet \vec{v} = 0$, and therefore the angle between the vectors is $\frac{\pi}{2}$.

Example

Find all vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both

$$\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Example

Find all vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both

$$\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution. There are infinitely many such vectors.

Example

Find all vectors $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ orthogonal to both

$$\vec{u} = \begin{bmatrix} -1 \\ -3 \\ 2 \end{bmatrix} \text{ and } \vec{w} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

Solution. There are infinitely many such vectors.
Since $\vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{w}$,

$$\begin{aligned} \vec{v} \bullet \vec{u} &= -x - 3y + 2z = 0 \\ \vec{v} \bullet \vec{w} &= y + z = 0 \end{aligned}$$

This is a homogeneous system of two linear equation in three variables.

$$\left[\begin{array}{rrr|r} -1 & -3 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right]$$

Example (continued)

$$\left[\begin{array}{rrr|r} 1 & 0 & -5 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \text{ implies that } \vec{v} = \left[\begin{array}{r} 5t \\ -t \\ t \end{array}\right] \text{ for } t \in \mathbb{R}.$$

Therefore, $\vec{v} = t \left[\begin{array}{r} 5 \\ -1 \\ 1 \end{array}\right]$ for all $t \in \mathbb{R}$.

Example

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

Example

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

Solution.

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

Example

Are $A(4, -7, 9)$, $B(6, 4, 4)$ and $C(7, 10, -6)$ the vertices of a right angle triangle?

Solution.

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

- $\overrightarrow{AB} \bullet \overrightarrow{AC} = 6 + 187 + 75 \neq 0$.

Example

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

Solution.

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

- $\overrightarrow{AB} \bullet \overrightarrow{AC} = 6 + 187 + 75 \neq 0.$
- $\overrightarrow{BA} \bullet \overrightarrow{BC} = (-\overrightarrow{AB}) \bullet \overrightarrow{BC} = -2 - 66 - 50 \neq 0.$

Example

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

Solution.

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

- $\overrightarrow{AB} \bullet \overrightarrow{AC} = 6 + 187 + 75 \neq 0.$
- $\overrightarrow{BA} \bullet \overrightarrow{BC} = (-\overrightarrow{AB}) \bullet \overrightarrow{BC} = -2 - 66 - 50 \neq 0.$
- $\overrightarrow{CA} \bullet \overrightarrow{CB} = (-\overrightarrow{AC}) \bullet (-\overrightarrow{BC}) = \overrightarrow{AC} \bullet \overrightarrow{BC} = 3 + 102 + 150 \neq 0.$

Example

Are $A(4,-7,9)$, $B(6,4,4)$ and $C(7,10,-6)$ the vertices of a right angle triangle?

Solution.

$$\overrightarrow{AB} = \begin{bmatrix} 2 \\ 11 \\ -5 \end{bmatrix}, \overrightarrow{AC} = \begin{bmatrix} 3 \\ 17 \\ -15 \end{bmatrix}, \overrightarrow{BC} = \begin{bmatrix} 1 \\ 6 \\ -10 \end{bmatrix}$$

- $\overrightarrow{AB} \bullet \overrightarrow{AC} = 6 + 187 + 75 \neq 0.$
- $\overrightarrow{BA} \bullet \overrightarrow{BC} = (-\overrightarrow{AB}) \bullet \overrightarrow{BC} = -2 - 66 - 50 \neq 0.$
- $\overrightarrow{CA} \bullet \overrightarrow{CB} = (-\overrightarrow{AC}) \bullet (-\overrightarrow{BC}) = \overrightarrow{AC} \bullet \overrightarrow{BC} = 3 + 102 + 150 \neq 0.$

None of the angles is $\frac{\pi}{2}$, and therefore the triangle is not a right angle triangle.

Work through §4.2 Example 4 yourselves.

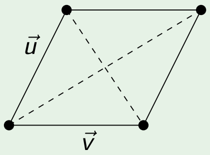
Example (§4.2 Example 5)

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Example (§4.2 Example 5)

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Solution.



Define the parallelogram (rhombus) by vectors $\vec{u}$ and $\vec{v}$.

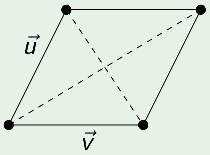
Then the diagonals are $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

Show that $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular.

Example (§4.2 Example 5)

A rhombus is a parallelogram with sides of equal length. Prove that the diagonals of a rhombus are perpendicular.

Solution.



Define the parallelogram (rhombus) by vectors $\vec{u}$ and $\vec{v}$.

Then the diagonals are $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$.

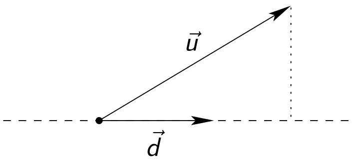
Show that $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are perpendicular.

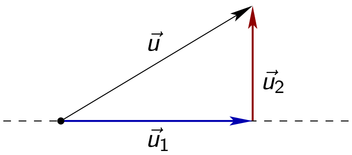
$$\begin{aligned}
(\vec{u}+\vec{v})\bullet(\vec{u}-\vec{v}) &= \vec{u}\bullet\vec{u}-\vec{u}\bullet\vec{v}+\vec{v}\bullet\vec{u}-\vec{v}\bullet\vec{v} \\
&= ||\vec{u}||^2-\vec{u}\bullet\vec{v}+\vec{u}\bullet\vec{v}-||\vec{v}||^2 \\
&= ||\vec{u}||^2-||\vec{v}||^2 \\
&= 0, \text{ since } ||\vec{u}|| = ||\vec{v}||.
\end{aligned}$$

Therefore, the diagonals are perpendicular.

Projections

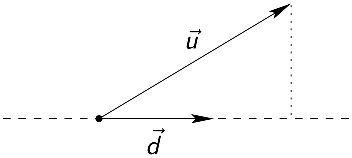
Given nonzero vectors $\vec{u}$ and $\vec{d}$, express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.

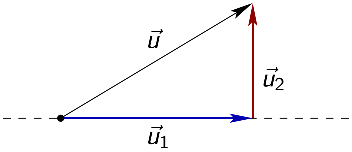




Projections

Given nonzero vectors $\vec{u}$ and $\vec{d}$, express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.





$\vec{u}_1$ is the projection of $\vec{u}$ onto $\vec{d}$, written $\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u}$.

Projections

Given nonzero vectors $\vec{u}$ and $\vec{d}$, express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.

$\vec{u}_1$ is the projection of $\vec{u}$ onto $\vec{d}$, written $\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u}$.

Since $\vec{u}_1$ is parallel to $\vec{d}$, $\vec{u}_1 = t\vec{d}$ for some $t \in \mathbb{R}$.

Projections

Given nonzero vectors $\vec{u}$ and $\vec{d}$, express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.

$\vec{u}_1$ is the projection of $\vec{u}$ onto $\vec{d}$, written $\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u}$.

Since $\vec{u}_1$ is parallel to $\vec{d}$, $\vec{u}_1 = t\vec{d}$ for some $t \in \mathbb{R}$.

Furthermore, $\vec{u}_2 = \vec{u} - \vec{u}_1$, so:

$$0 = \vec{u}_2 \bullet \vec{d} = (\vec{u} - \vec{u}_1) \bullet \vec{d} = (\vec{u} - t\vec{d}) \bullet \vec{d} = \vec{u} \bullet \vec{d} - t(\vec{d} \bullet \vec{d})$$

Projections

Given nonzero vectors $\vec{u}$ and $\vec{d}$, express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.

$\vec{u}_1$ is the projection of $\vec{u}$ onto $\vec{d}$, written $\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u}$.

Since $\vec{u}_1$ is parallel to $\vec{d}$, $\vec{u}_1 = t\vec{d}$ for some $t \in \mathbb{R}$.

Furthermore, $\vec{u}_2 = \vec{u} - \vec{u}_1$, so:

$$0 = \vec{u}_2 \bullet \vec{d} = (\vec{u} - \vec{u}_1) \bullet \vec{d} = (\vec{u} - t\vec{d}) \bullet \vec{d} = \vec{u} \bullet \vec{d} - t(\vec{d} \bullet \vec{d})$$

Hence since $\vec{d} \neq \vec{0}$, we get $t = \frac{\vec{u} \bullet \vec{d}}{||\vec{d}||^2}$,

Projections

Given nonzero vectors $\vec{u}$ and $\vec{d}$, express $\vec{u}$ as a sum $\vec{u} = \vec{u}_1 + \vec{u}_2$, where $\vec{u}_1$ is parallel to $\vec{d}$ and $\vec{u}_2$ is orthogonal to $\vec{d}$.

$\vec{u}_1$ is the projection of $\vec{u}$ onto $\vec{d}$, written $\vec{u}_1 = \text{proj}_{\vec{d}}\vec{u}$.

Since $\vec{u}_1$ is parallel to $\vec{d}$, $\vec{u}_1 = t\vec{d}$ for some $t \in \mathbb{R}$.

Furthermore, $\vec{u}_2 = \vec{u} - \vec{u}_1$, so:

$$0 = \vec{u}_2 \bullet \vec{d} = (\vec{u} - \vec{u}_1) \bullet \vec{d} = (\vec{u} - t\vec{d}) \bullet \vec{d} = \vec{u} \bullet \vec{d} - t(\vec{d} \bullet \vec{d})$$

Hence since $\vec{d} \neq \vec{0}$, we get $t = \frac{\vec{u} \bullet \vec{d}}{||\vec{d}||^2}$, and therefore

$$\vec{u}_1 = \left(\frac{\vec{u} \bullet \vec{d}}{||\vec{d}||^2} \right) \vec{d}.$$

Theorem (§4.2 Theorem 4)

Let $\vec{u}$ and $\vec{d}$ be vectors with $\vec{d} \neq \vec{0}$.

Theorem (§4.2 Theorem 4)

Let $\vec{u}$ and $\vec{d}$ be vectors with $\vec{d} \neq \vec{0}$.

1. $$proj_{\vec{d}}\vec{u} = \left(\frac{\vec{u} \bullet \vec{d}}{||\vec{d}||^2} \right) \vec{d}.$$

Theorem (§4.2 Theorem 4)

Let $\vec{u}$ and $\vec{d}$ be vectors with $\vec{d} \neq \vec{0}$.

1. $$proj_{\vec{d}}\vec{u} = \left(\frac{\vec{u} \bullet \vec{d}}{||\vec{d}||^2}\right)\vec{d}.$$

2. $$\vec{u} - \left(\frac{\vec{u} \bullet \vec{d}}{||\vec{d}||^2}\right)\vec{d}$$

   is orthogonal to $\vec{d}$.

Example

Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Find vectors $\vec{u}_1$ and $\vec{u}_2$ so that $\vec{u} = \vec{u}_1 + \vec{u}_2$, with $\vec{u}_1$ parallel to $\vec{v}$ and $\vec{u}_2$ orthogonal to $\vec{v}$.

Example

Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Find vectors $\vec{u}_1$ and $\vec{u}_2$ so that $\vec{u} = \vec{u}_1 + \vec{u}_2$, with $\vec{u}_1$ parallel to $\vec{v}$ and $\vec{u}_2$ orthogonal to $\vec{v}$.

Solution.

$$\vec{u}_1 = \text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \bullet \vec{v}}{||\vec{v}||^2}\right)\vec{v} = \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 15/11 \\ 5/11 \\ -5/11 \end{bmatrix}.$$

Example

Let $\vec{u} = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix}$. Find vectors $\vec{u}_1$ and $\vec{u}_2$ so that $\vec{u} = \vec{u}_1 + \vec{u}_2$, with $\vec{u}_1$ parallel to $\vec{v}$ and $\vec{u}_2$ orthogonal to $\vec{v}$.

Solution.

$$\vec{u}_1 = \text{proj}_{\vec{v}}\vec{u} = \left(\frac{\vec{u} \bullet \vec{v}}{||\vec{v}||^2}\right)\vec{v} = \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 15/11 \\ 5/11 \\ -5/11 \end{bmatrix}.$$

$$\vec{u}_2 = \vec{u} - \vec{u}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} - \frac{5}{11}\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 7 \\ -16 \\ 5 \end{bmatrix} = \begin{bmatrix} 7/11 \\ -16/11 \\ 5/11 \end{bmatrix}.$$

Distance from a Point to a Line

Example

Let $P(3, 2, -1)$ be a point in $\mathbb{R}^3$ and L a line with equation

$$\left[\begin{array}{c} x \\ y \\ z \end{array}\right] = \left[\begin{array}{c} 2 \\ 1 \\ 3 \end{array}\right] + t\left[\begin{array}{r} 3 \\ -1 \\ -2 \end{array}\right].$$

Find the shortest distance from P to L, and find the point Q on L that is closest to P.

Distance from a Point to a Line

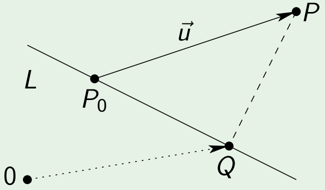
Example

Let $P(3,2,-1)$ be a point in $\mathbb{R}^3$ and L a line with equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} + t \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}.$$

Find the shortest distance from P to L, and find the point Q on L that is closest to P.

Solution.



Let $P_0 = P_0(2,1,3)$ be a point on L, and let $\vec{d} = \begin{bmatrix} 3 & -1 & -2 \end{bmatrix}^T$. Then $\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P}$, $\overrightarrow{0Q} = \overrightarrow{0P_0} + \overrightarrow{P_0Q}$, and the shortest distance from P to L is the length of $\overrightarrow{QP}$, where $\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q}$.

Example (continued)

$\overrightarrow{P_0P} = \left[\begin{array}{ccc} 1 & 1 & -4 \end{array} \right]^T$, $\vec{d} = \left[\begin{array}{ccc} 3 & -1 & -2 \end{array} \right]^T$.

$$\overrightarrow{P_0Q} = \text{proj}_{\vec{d}} \overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{d}}{||\vec{d}||^2} \right) \vec{d} = \frac{10}{14} \left[\begin{array}{r} 3 \\ -1 \\ -2 \end{array} \right] = \frac{1}{7} \left[\begin{array}{r} 15 \\ -5 \\ -10 \end{array} \right].$$

Example (continued)

$\overrightarrow{P_0P} = \left[\begin{array}{ccc} 1 & 1 & -4 \end{array}\right]^T$, $\vec{d} = \left[\begin{array}{ccc} 3 & -1 & -2 \end{array}\right]^T$.

$$\overrightarrow{P_0Q} = \text{proj}_{\vec{d}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{d}}{||\vec{d}||^2}\right)\vec{d} = \frac{10}{14}\left[\begin{array}{r} 3 \\ -1 \\ -2 \end{array}\right] = \frac{1}{7}\left[\begin{array}{r} 15 \\ -5 \\ -10 \end{array}\right].$$

Therefore,

$$\overrightarrow{0Q} = \left[\begin{array}{c} 2 \\ 1 \\ 3 \end{array}\right] + \frac{1}{7}\left[\begin{array}{r} 15 \\ -5 \\ -10 \end{array}\right] = \frac{1}{7}\left[\begin{array}{r} 29 \\ 2 \\ 11 \end{array}\right],$$

so $Q = Q\left(\frac{29}{7}, \frac{2}{7}, \frac{11}{7}\right)$.

Example (continued)

Finally, the shortest distance from $P(3,2,-1)$ to L is the length of $\overrightarrow{QP}$, where

$$\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} -4 \\ 6 \\ -9 \end{bmatrix}.$$

Example (continued)

Finally, the shortest distance from $P(3,2,-1)$ to L is the length of $\overrightarrow{QP}$, where

$$\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} -4 \\ 6 \\ -9 \end{bmatrix}.$$

Therefore the shortest distance from P to L is

$$||\overrightarrow{QP}|| = \frac{2}{7}\sqrt{(-4)^2 + 6^2 + (-9)^2} = \frac{2}{7}\sqrt{133}.$$

Example (continued)

Finally, the shortest distance from $P(3, 2, -1)$ to L is the length of $\overrightarrow{QP}$, where

$$\overrightarrow{QP} = \overrightarrow{P_0P} - \overrightarrow{P_0Q} = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 15 \\ -5 \\ -10 \end{bmatrix} = \frac{2}{7}\begin{bmatrix} -4 \\ 6 \\ -9 \end{bmatrix}.$$

Therefore the shortest distance from P to L is

$$||\overrightarrow{QP}|| = \frac{2}{7}\sqrt{(-4)^2 + 6^2 + (-9)^2} = \frac{2}{7}\sqrt{133}.$$

§**4.2 Example 8** is similar.

Equations of Planes

Given a point P_0 and a nonzero vector $\vec{n}$, there is a unique plane containing P_0 and orthogonal to $\vec{n}$.

Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \bullet \vec{v} = 0$ for every vector $\vec{v}$ in the plane.

Equations of Planes

Given a point P_0 and a nonzero vector $\vec{n}$, there is a unique plane containing P_0 and orthogonal to $\vec{n}$.

Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \bullet \vec{v} = 0$ for every vector $\vec{v}$ in the plane.

Consider a plane containing a point P_0 and orthogonal to vector $\vec{n}$, and let P be an arbitrary point on this plane.

Equations of Planes

Given a point P_0 and a nonzero vector $\vec{n}$, there is a unique plane containing P_0 and orthogonal to $\vec{n}$.

Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \bullet \vec{v} = 0$ for every vector $\vec{v}$ in the plane.

Consider a plane containing a point P_0 and orthogonal to vector $\vec{n}$, and let P be an arbitrary point on this plane.
Then

$$\vec{n} \bullet \overrightarrow{P_0P} = 0,$$

Equations of Planes

Given a point P_0 and a nonzero vector $\vec{n}$, there is a unique plane containing P_0 and orthogonal to $\vec{n}$.

Definition

A nonzero vector $\vec{n}$ is a normal vector to a plane if and only if $\vec{n} \bullet \vec{v} = 0$ for every vector $\vec{v}$ in the plane.

Consider a plane containing a point P_0 and orthogonal to vector $\vec{n}$, and let P be an arbitrary point on this plane.
Then

$$\vec{n} \bullet \overrightarrow{P_0P} = 0,$$

or, equivalently,

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0,$$

and is a vector equation of the plane.

The vector equation

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

can also be written as

$$\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}.$$

The vector equation

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

can also be written as

$$\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}.$$

Now suppose $P_0 = P_0(x_0, y_0, z_0)$, $P = P(x, y, z)$, and $\vec{n} = \left[\begin{array}{ccc} a & b & c \end{array} \right]^T$.

The vector equation

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

can also be written as

$$\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}.$$

Now suppose $P_0 = P_0(x_0, y_0, z_0)$, $P = P(x, y, z)$, and $\vec{n} = \begin{bmatrix} a & b & c \end{bmatrix}^T$. Then the previous equation becomes

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix},$$

The vector equation

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

can also be written as

$$\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}.$$

Now suppose $P_0 = P_0(x_0, y_0, z_0)$, $P = P(x, y, z)$, and $\vec{n} = \begin{bmatrix} a & b & c \end{bmatrix}^T$.
Then the previous equation becomes

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \bullet \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix},$$

so

$$ax + by + cz = ax_0 + by_0 + cz_0,$$

where $d = ax_0 + by_0 + cz_0$ is simply a scalar.

The vector equation

$$\vec{n} \bullet (\overrightarrow{0P} - \overrightarrow{0P_0}) = 0$$

can also be written as

$$\vec{n} \bullet \overrightarrow{0P} = \vec{n} \bullet \overrightarrow{0P_0}.$$

Now suppose $P_0 = P_0(x_0, y_0, z_0)$, $P = P(x, y, z)$, and $\vec{n} = \left[\begin{array}{ccc} a & b & c \end{array} \right]^T$.
Then the previous equation becomes

$$\left[\begin{array}{c} a \\ b \\ c \end{array} \right] \bullet \left[\begin{array}{c} x \\ y \\ z \end{array} \right] = \left[\begin{array}{c} a \\ b \\ c \end{array} \right] \bullet \left[\begin{array}{c} x_0 \\ y_0 \\ z_0 \end{array} \right],$$

so

$$ax + by + cz = ax_0 + by_0 + cz_0,$$

where $d = ax_0 + by_0 + cz_0$ is simply a scalar.
A scalar equation of the plane has the form

$$ax + by + cz = d, \text{ where } a, b, c, d \in \mathbb{R}.$$

Example

Find an equation of the plane containing $P_0(1, -1, 0)$ and orthogonal to $\vec{n} = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}^T$.

Example

Find an equation of the plane containing $P_0(1, -1, 0)$ and orthogonal to $\vec{n} = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}^T$.

Solution.

A vector equation of this plane is

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} x-1 \\ y+1 \\ z \end{bmatrix} = 0.$$

Example

Find an equation of the plane containing $P_0(1, -1, 0)$ and orthogonal to $\vec{n} = \begin{bmatrix} -3 & 5 & 2 \end{bmatrix}^T$.

Solution.

A vector equation of this plane is

$$\begin{bmatrix} -3 \\ 5 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} x-1 \\ y+1 \\ z \end{bmatrix} = 0.$$

A scalar equation of this plane is

$$-3x + 5y + 2z = -3(1) + 5(-1) + 2(0) = -8,$$

i.e., the plane has scalar equation

$$-3x + 5y + 2z = -8.$$

§**4.2 Example 11**: Two solutions to the problem of finding the shortest distance from a point to a plane.

§**4.2 Example 11**: Two solutions to the problem of finding the shortest distance from a point to a plane.

Example

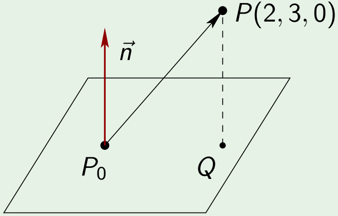
Find the shortest distance from the point $P(2, 3, 0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

§**4.2 Example 11**: Two solutions to the problem of finding the shortest distance from a point to a plane.

Example

Find the shortest distance from the point $P(2, 3, 0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

Solution (like the first solution to §4.2 Example 11).



Pick an arbitrary point P_0 on the plane.

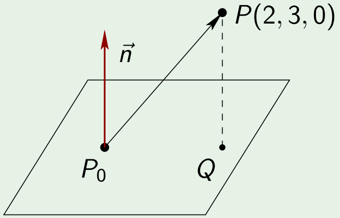
Then $\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P}$,
$||\overrightarrow{QP}||$ is the shortest distance,
and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

§**4.2 Example 11**: Two solutions to the problem of finding the shortest distance from a point to a plane.

Example

Find the shortest distance from the point $P(2,3,0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

Solution (like the first solution to §4.2 Example 11).



Pick an arbitrary point P_0 on the plane.

Then $\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P}$,
$||\overrightarrow{QP}||$ is the shortest distance,
and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

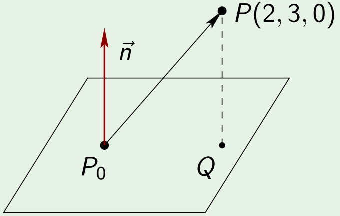
$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T$.

§**4.2 Example 11**: Two solutions to the problem of finding the shortest distance from a point to a plane.

Example

Find the shortest distance from the point $P(2,3,0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

Solution (like the first solution to §4.2 Example 11).



Pick an arbitrary point P_0 on the plane.

Then $\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P}$,
$||\overrightarrow{QP}||$ is the shortest distance,
and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

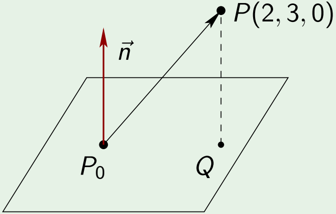
$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T$. Choose $P_0 = P_0(0,0,-1)$.

§4.2 **Example 11**: Two solutions to the problem of finding the shortest distance from a point to a plane.

Example

Find the shortest distance from the point $P(2,3,0)$ to the plane with equation $5x + y + z = -1$, and find the point Q on the plane that is closest to P.

Solution (like the first solution to §4.2 Example 11).

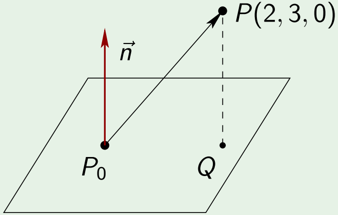


Pick an arbitrary point P_0 on the plane.

Then $\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P}$,
$||\overrightarrow{QP}||$ is the shortest distance,
and $\overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP}$.

$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T$. Choose $P_0 = P_0(0,0,-1)$.
Then $\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T$.

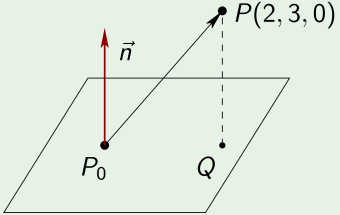
Example (continued)



$$\overrightarrow{P_0P} = \left[\begin{array}{ccc} 2 & 3 & 1 \end{array}\right]^T.$$

$$\vec{n} = \left[\begin{array}{ccc} 5 & 1 & 1 \end{array}\right]^T.$$

Example (continued)

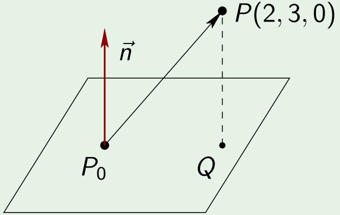


$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T.$$

$$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

$$\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{n}}{||\vec{n}||^2} \right) \vec{n} = \frac{14}{27} \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

Example (continued)



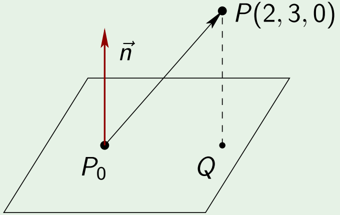
$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T.$$

$$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

$$\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{n}}{||\vec{n}||^2} \right) \vec{n} = \frac{14}{27} \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

Since $||\overrightarrow{QP}|| = \frac{14}{27}\sqrt{27} = \frac{14\sqrt{3}}{9}$, the shortest distance from P to the plane is $\frac{14\sqrt{3}}{9}$.

Example (continued)



$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T.$$

$$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

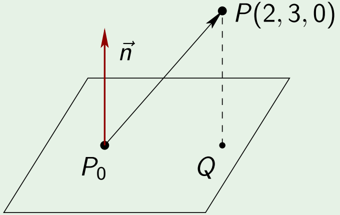
$$\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{n}}{||\vec{n}||^2}\right)\vec{n} = \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

Since $||\overrightarrow{QP}|| = \frac{14}{27}\sqrt{27} = \frac{14\sqrt{3}}{9}$, the shortest distance from P to the plane is $\frac{14\sqrt{3}}{9}$.

To find Q, we have

$$\begin{aligned} \overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP} &= \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}^T - \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T \\ &= \frac{1}{27}\begin{bmatrix} -16 & 67 & -14 \end{bmatrix}^T. \end{aligned}$$

Example (continued)



$$\overrightarrow{P_0P} = \begin{bmatrix} 2 & 3 & 1 \end{bmatrix}^T.$$

$$\vec{n} = \begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

$$\overrightarrow{QP} = \text{proj}_{\vec{n}}\overrightarrow{P_0P} = \left(\frac{\overrightarrow{P_0P} \bullet \vec{n}}{||\vec{n}||^2}\right)\vec{n} = \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T.$$

Since $||\overrightarrow{QP}|| = \frac{14}{27}\sqrt{27} = \frac{14\sqrt{3}}{9}$, the shortest distance from P to the plane is $\frac{14\sqrt{3}}{9}$.

To find Q, we have

$$\begin{aligned} \overrightarrow{0Q} = \overrightarrow{0P} - \overrightarrow{QP} &= \begin{bmatrix} 2 & 3 & 0 \end{bmatrix}^T - \frac{14}{27}\begin{bmatrix} 5 & 1 & 1 \end{bmatrix}^T \\ &= \frac{1}{27}\begin{bmatrix} -16 & 67 & -14 \end{bmatrix}^T. \end{aligned}$$

Therefore $Q = Q\left(-\frac{16}{27}, \frac{67}{27}, -\frac{14}{27}\right)$.

The Cross Product

Definition

Let $\vec{u} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$. Then

$$\vec{u} \times \vec{v} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

The Cross Product

Definition

Let $\vec{u} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$. Then

$$\vec{u} \times \vec{v} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

Note. $\vec{u} \times \vec{v}$ is a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$.

The Cross Product

Definition

Let $\vec{u} = \begin{bmatrix} x_1 & y_1 & z_1 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} x_2 & y_2 & z_2 \end{bmatrix}^T$. Then

$$\vec{u} \times \vec{v} = \begin{bmatrix} y_1 z_2 - z_1 y_2 \\ -(x_1 z_2 - z_1 x_2) \\ x_1 y_2 - y_1 x_2 \end{bmatrix}.$$

Note. $\vec{u} \times \vec{v}$ is a vector that is orthogonal to both $\vec{u}$ and $\vec{v}$.

A mnemonic device:

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & x_1 & x_2 \\ \vec{j} & y_1 & y_2 \\ \vec{k} & z_1 & z_2 \end{vmatrix}, \text{ where } \vec{i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \vec{j} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \vec{k} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Theorem (§4.2 Theorem 5)

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$.

Theorem (§4.2 Theorem 5)

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$.

1. *$\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.*

Theorem (§4.2 Theorem 5)

Let $\vec{u}, \vec{v} \in \mathbb{R}^3$.

1. $\vec{u} \times \vec{v}$ *is orthogonal to both* $\vec{u}$ *and* $\vec{v}$.
2. *If* $\vec{u}$ *and* $\vec{v}$ *are both nonzero, then* $\vec{u} \times \vec{v} = \vec{0}$ *if and only if* $\vec{u}$ *and* $\vec{v}$ *are parallel.*

Example

Find all vectors orthogonal to both $\vec{u} = \begin{bmatrix} -1 & -3 & 2 \end{bmatrix}^T$ and $\vec{v} = \begin{bmatrix} 0 & 1 & 1 \end{bmatrix}^T$.

(We previously solved this using the dot product.)

Example

Find all vectors orthogonal to both $\vec{u} = \left[\begin{array}{ccc} -1 & -3 & 2 \end{array}\right]^T$ and $\vec{v} = \left[\begin{array}{ccc} 0 & 1 & 1 \end{array}\right]^T$.

(We previously solved this using the dot product.)

Solution.

$$\vec{u} \times \vec{v} = \left|\begin{array}{ccc} \vec{i} & -1 & 0 \\ \vec{j} & -3 & 1 \\ \vec{k} & 2 & 1 \end{array}\right| = -5\vec{i} + \vec{j} - \vec{k} = \left[\begin{array}{r} -5 \\ 1 \\ -1 \end{array}\right].$$

Example

Find all vectors orthogonal to both $\vec{u} = \left[\begin{array}{ccc} -1 & -3 & 2 \end{array}\right]^T$ and $\vec{v} = \left[\begin{array}{ccc} 0 & 1 & 1 \end{array}\right]^T$.

(We previously solved this using the dot product.)

Solution.

$$\vec{u} \times \vec{v} = \left|\begin{array}{ccc} \vec{i} & -1 & 0 \\ \vec{j} & -3 & 1 \\ \vec{k} & 2 & 1 \end{array}\right| = -5\vec{i} + \vec{j} - \vec{k} = \left[\begin{array}{r} -5 \\ 1 \\ -1 \end{array}\right].$$

Any scalar multiple of $\vec{u} \times \vec{v}$ is also orthogonal to both $\vec{u}$ and $\vec{v}$, so

$$t\left[\begin{array}{r} -5 \\ 1 \\ -1 \end{array}\right], t \in \mathbb{R},$$

gives all vectors orthogonal to both $\vec{u}$ and $\vec{v}$.

(Compare this with our earlier answer.)

§**4.2 Example 13** shows how to find an equation of a plane that contains three non-colinear points.

§**4.2 Example 14** shows how to find the shortest distance between skew lines, i.e., lines that are not parallel and do not intersect.

Distance between skew lines

Example

Given two lines

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

A. Find the shortest distance between L_1 and L_2.

B. Find the shortest distance between L_1 and L_2, and find the points P on L_1 and Q on L_2 that are closest together.

Distance between skew lines

Example

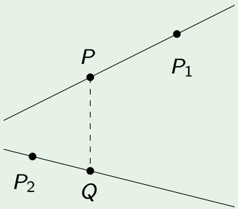
Given two lines

$$L_1 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + s \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \text{ and } L_2 : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix},$$

A. Find the shortest distance between L_1 and L_2.

B. Find the shortest distance between L_1 and L_2, and find the points P on L_1 and Q on L_2 that are closest together.

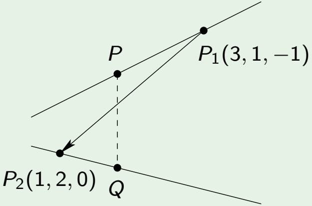
Solution A.



Choose $P_1(3, 1, -1)$ on L_1 and $P_2(1, 2, 0)$ on L_2.

Let $\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ and $\vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ denote direction vectors for L_1 and L_2, respectively.

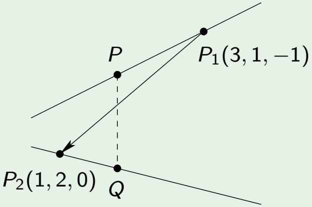
Example (continued)



$$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The shortest distance between L_1 and L_2 is the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{n} = \vec{d}_1 \times \vec{d}_2$.

Example (continued)

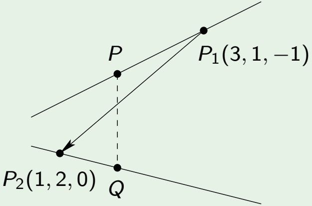


$$\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The shortest distance between L_1 and L_2 is the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{n} = \vec{d_1} \times \vec{d_2}$.

$$\overrightarrow{P_1P_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

Example (continued)



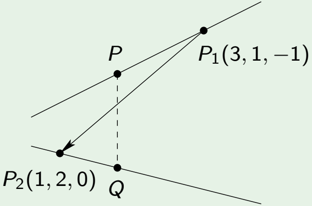
$$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The shortest distance between L_1 and L_2 is the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{n} = \vec{d}_1 \times \vec{d}_2$.

$$\overrightarrow{P_1P_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\vec{n}}\overrightarrow{P_1P_2} = \left(\frac{\overrightarrow{P_1P_2} \bullet \vec{n}}{||\vec{n}||^2} \right) \vec{n}, \text{ and } ||\text{proj}_{\vec{n}}\overrightarrow{P_1P_2}|| = \frac{|\overrightarrow{P_1P_2} \bullet \vec{n}|}{||\vec{n}||}.$$

Example (continued)



$$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

The shortest distance between L_1 and L_2 is the length of the projection of $\overrightarrow{P_1P_2}$ onto $\vec{n} = \vec{d}_1 \times \vec{d}_2$.

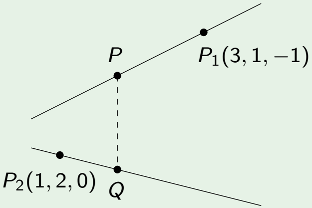
$$\overrightarrow{P_1P_2} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \text{ and } \vec{n} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix}$$

$$\text{proj}_{\vec{n}}\overrightarrow{P_1P_2} = \left(\frac{\overrightarrow{P_1P_2} \bullet \vec{n}}{||\vec{n}||^2} \right) \vec{n}, \text{ and } ||\text{proj}_{\vec{n}}\overrightarrow{P_1P_2}|| = \frac{|\overrightarrow{P_1P_2} \bullet \vec{n}|}{||\vec{n}||}.$$

Therefore, the shortest distance between L_1 and L_2 is $\frac{|-8|}{\sqrt{14}} = \frac{4}{7}\sqrt{14}$.

Example (continued)

Solution B.



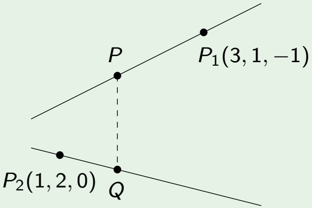
$$\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$$

$$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix} \text{ for some } s \in \mathbb{R};$$

$$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix} \text{ for some } t \in \mathbb{R}.$$

Example (continued)

Solution B.



$\vec{d}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d}_2 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$

$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix}$ for some $s \in \mathbb{R}$;

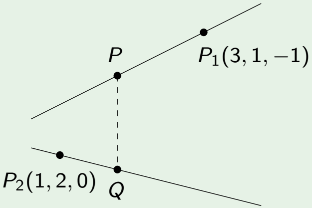
$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix}$ for some $t \in \mathbb{R}$.

Now $\overrightarrow{PQ} = \begin{bmatrix} -2-s+t & 1-s & 1+s+2t \end{bmatrix}^T$ is orthogonal to both L_1 and L_2, so

$$\overrightarrow{PQ} \bullet \vec{d}_1 = 0 \text{ and } \overrightarrow{PQ} \bullet \vec{d}_2 = 0,$$

Example (continued)

Solution B.



$$\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \ \vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$$

$$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix} \text{ for some } s \in \mathbb{R};$$

$$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix} \text{ for some } t \in \mathbb{R}.$$

Now $\overrightarrow{PQ} = \begin{bmatrix} -2-s+t & 1-s & 1+s+2t \end{bmatrix}^T$ is orthogonal to both L_1 and L_2, so

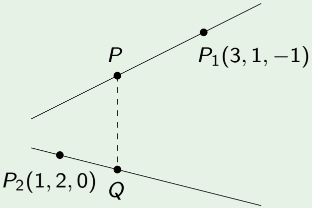
$$\overrightarrow{PQ} \bullet \vec{d_1} = 0 \text{ and } \overrightarrow{PQ} \bullet \vec{d_2} = 0,$$

i.e.,

$$\begin{aligned} -2-3s-t &= 0 \\ s+5t &= 0. \end{aligned}$$

Example (continued)

Solution B.



$$\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$$

$$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix} \text{ for some } s \in \mathbb{R};$$

$$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix} \text{ for some } t \in \mathbb{R}.$$

Now $\overrightarrow{PQ} = \begin{bmatrix} -2-s+t & 1-s & 1+s+2t \end{bmatrix}^T$ is orthogonal to both L_1 and L_2, so

$$\overrightarrow{PQ} \bullet \vec{d_1} = 0 \text{ and } \overrightarrow{PQ} \bullet \vec{d_2} = 0,$$

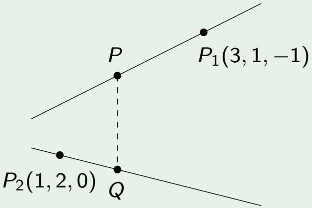
i.e.,

$$\begin{aligned} -2-3s-t &= 0 \\ s+5t &= 0. \end{aligned}$$

This system has unique solution $s = -\frac{5}{7}$ and $t = \frac{1}{7}$.

Example (continued)

Solution B.



$$\vec{d_1} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \vec{d_2} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix};$$

$$\overrightarrow{0P} = \begin{bmatrix} 3+s \\ 1+s \\ -1-s \end{bmatrix} \text{ for some } s \in \mathbb{R};$$

$$\overrightarrow{0Q} = \begin{bmatrix} 1+t \\ 2 \\ 2t \end{bmatrix} \text{ for some } t \in \mathbb{R}.$$

Now $\overrightarrow{PQ} = \begin{bmatrix} -2-s+t & 1-s & 1+s+2t \end{bmatrix}^T$ is orthogonal to both L_1 and L_2, so

$$\overrightarrow{PQ} \bullet \vec{d_1} = 0 \text{ and } \overrightarrow{PQ} \bullet \vec{d_2} = 0,$$

i.e.,

$$\begin{aligned} -2-3s-t &= 0 \\ s+5t &= 0. \end{aligned}$$

This system has unique solution $s = -\frac{5}{7}$ and $t = \frac{1}{7}$. Therefore,

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \text{ and } Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right).$$

Example (continued)

The shortest distance between L_1 and L_2 is $||\overrightarrow{PQ}||$. Since

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \text{ and } Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right),$$

Example (continued)

The shortest distance between L_1 and L_2 is $||\overrightarrow{PQ}||$. Since

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \text{ and } Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right),$$

$$\overrightarrow{PQ} = \frac{1}{7}\begin{bmatrix} 8 \\ 14 \\ 2 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 16 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -8 \\ 12 \\ 4 \end{bmatrix},$$

Example (continued)

The shortest distance between L_1 and L_2 is $||\overrightarrow{PQ}||$. Since

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \text{ and } Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right),$$

$$\overrightarrow{PQ} = \frac{1}{7}\begin{bmatrix} 8 \\ 14 \\ 2 \end{bmatrix} - \frac{1}{7}\begin{bmatrix} 16 \\ 2 \\ -2 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} -8 \\ 12 \\ 4 \end{bmatrix},$$

and

$$||\overrightarrow{PQ}|| = \frac{1}{7}\sqrt{224} = \frac{4}{7}\sqrt{14}.$$

Example (continued)

The shortest distance between L_1 and L_2 is $||\overrightarrow{PQ}||$. Since

$$P = P\left(\frac{16}{7}, \frac{2}{7}, -\frac{2}{7}\right) \text{ and } Q = Q\left(\frac{8}{7}, 2, \frac{2}{7}\right),$$

$$\overrightarrow{PQ} = \frac{1}{7}\left[\begin{array}{r} 8 \\ 14 \\ 2 \end{array}\right] - \frac{1}{7}\left[\begin{array}{r} 16 \\ 2 \\ -2 \end{array}\right] = \frac{1}{7}\left[\begin{array}{r} -8 \\ 12 \\ 4 \end{array}\right],$$

and

$$||\overrightarrow{PQ}|| = \frac{1}{7}\sqrt{224} = \frac{4}{7}\sqrt{14}.$$

Therefore the shortest distance between L_1 and L_2 is $\frac{4}{7}\sqrt{14}$.