MATH 211 – Fall 2012

Lecture Notes

K. Seyffarth

Section 4.1

§4.1 – Vectors and Lines

Scalar quantities vs. Vector quantities

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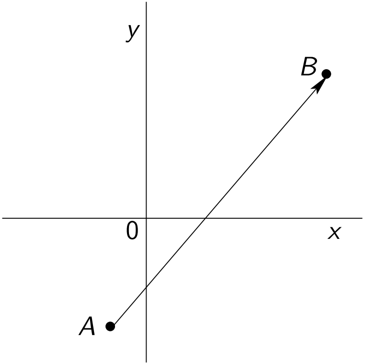
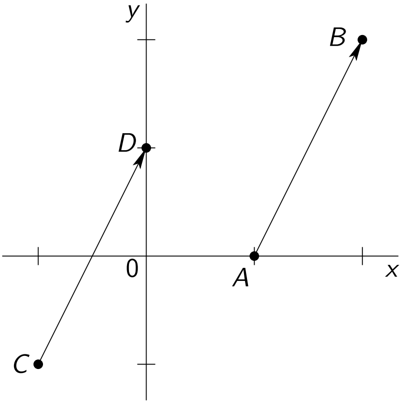
Two vector quantities are equal if and only if they have the same magnitude and direction.

Geometric Vectors

Let A and B be two points in $\mathbb{R}^2$ or $\mathbb{R}^3$.



- $\overrightarrow{AB}$ is the geometric vector from A to B.
- A is the tail of $\overrightarrow{AB}$.
- B is the tip of $\overrightarrow{AB}$.
- the magnitude of $\overrightarrow{AB}$ is its length, and is denoted $||\overrightarrow{AB}||$.



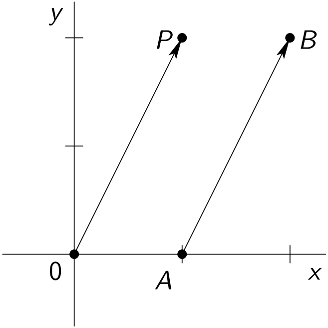
- $\overrightarrow{AB}$ is the vector from $A(1,0)$ to $B(2,2)$.
- $\overrightarrow{CD}$ is the vector from $C(-1,-1)$ to $D(0,1)$.

- $\overrightarrow{AB} = \overrightarrow{CD}$ because the vectors have the same length and direction.

Definition

A vector is in standard position if its tail is at the origin.

We co-ordinatize vectors by putting them in standard position, and then identifying them with their tips.



Thus $\overrightarrow{AB} = \overrightarrow{0P}$ where $P = P(1,2)$, and we write $\overrightarrow{0P} = \left[\begin{array}{c} 1 \\ 2 \end{array}\right] = \overrightarrow{AB}$.

$\overrightarrow{0P}$ is the position vector for $P(1,2)$.

More generally, if $P(x, y, z)$ is a point in $\mathbb{R}^3$, then $\overrightarrow{0P} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is the position vector for P.

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If we aren't concerned with the locations of the tail and tip, we simply write $\vec{v} = \left[\begin{array}{c} x \\ y \\ z \end{array}\right]$

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Let $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ *and* $\vec{w} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}$ *be vectors in* $\mathbb{R}^3$. *Then*

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1. $\vec{v} = \vec{w}$ *if and only if* $x = x_1$, $y = y_1$, *and* $z = z_1$.
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Analogous results hold for $\vec{v}, \vec{w} \in \mathbb{R}^2$, i.e.,

$$\vec{v} = \begin{bmatrix} x \\ y \end{bmatrix}, \vec{w} = \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}.$$

In this case, $||\vec{v}|| = \sqrt{x^2 + y^2}$.

Example

Let $\vec{p} = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$, $\vec{q} = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}$, and $-2\vec{q} = \begin{bmatrix} -6 \\ 2 \\ 4 \end{bmatrix}$,

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and

$$\begin{aligned} ||-2\vec{q}|| &= \sqrt{(-6)^2 + 2^2 + 4^2} \\ &= \sqrt{36 + 4 + 16} \\ &= \sqrt{56} = \sqrt{4 \times 14} \\ &= 2\sqrt{14} = 2||\vec{q}||. \end{aligned}$$

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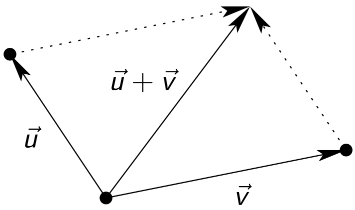

parallelogram law

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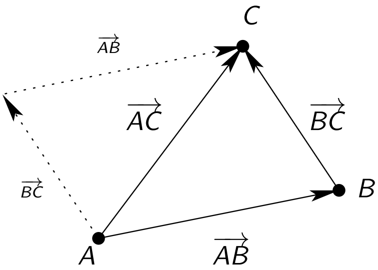
- addition: $\vec{u} + \vec{v}$ is represented by the matrix sum of the columns $\vec{u}$ and $\vec{v}$.

Tip-to-Tail Method for Vector Addition

For points A, B and C,

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}.$$



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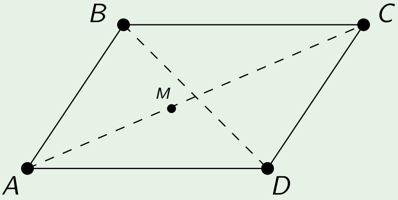

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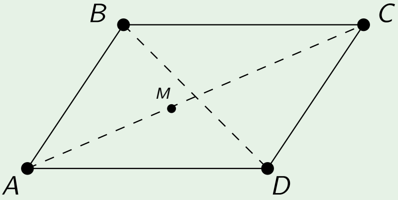

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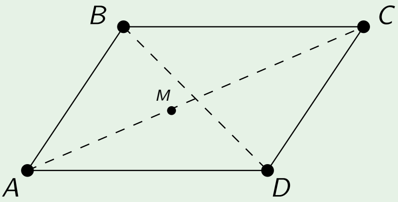

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Since $\overrightarrow{BM} = \overrightarrow{MD}$, these vectors have the same magnitude and direction, implying that M is the midpoint of $\overrightarrow{BD}$.

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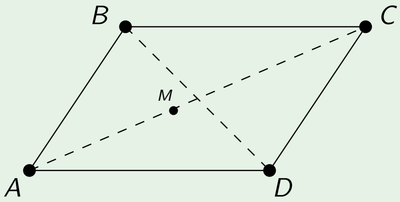

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Therefore, the diagonals of $ABCD$ bisect each other.

Vector Subtraction

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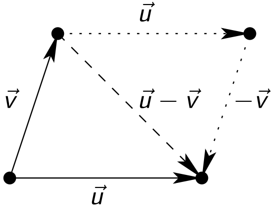

$\vec{u} - \vec{v} = \vec{u} + (-\vec{v})$ and is the diagonal from the tip of $\vec{v}$ to the tip of $\vec{u}$ in the parallelogram defined by $\vec{u}$ and $\vec{v}$.

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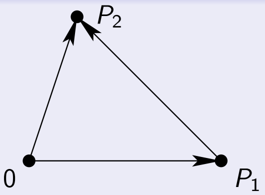
Proof.



$\overrightarrow{0P_1} + \overrightarrow{P_1P_2} = \overrightarrow{0P_2}$, so $\overrightarrow{P_1P_2} = \overrightarrow{0P_2} - \overrightarrow{0P_1}$, and the distance between P_1 and P_2 is $||\overrightarrow{P_1P_2}||$. □

Example

For $P(1,-1,3)$ and $Q(3,1,0)$

$$\overrightarrow{PQ} = \begin{bmatrix} 3-1 \\ 1-(-1) \\ 0-3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$$

and the distance between P and Q is $||\overrightarrow{PQ}|| = \sqrt{2^2+2^2+(-3)^2} = \sqrt{17}$.

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A unit vector is a vector of length one.

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$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, are examples of unit vectors.

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$$||\vec{u}|| = \frac{1}{\sqrt{14}}||\vec{v}|| = \frac{1}{\sqrt{14}}\sqrt{14} = 1.$$

Example (§4.1 Example 4)

If $\vec{v} \neq \vec{0}$, then

$$\frac{1}{||\vec{v}||}\vec{v}$$

is a unit vector in the same direction as $\vec{v}$.

Example

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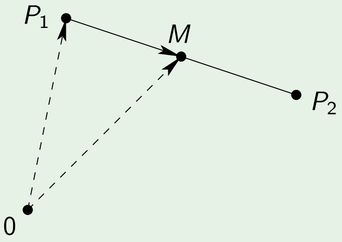

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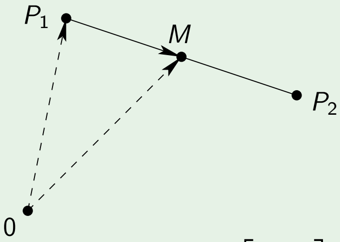

$$\overrightarrow{0M} = \overrightarrow{0P_1} + \overrightarrow{P_1M} = \overrightarrow{0P_1} + \frac{1}{2}\overrightarrow{P_1P_2} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 6 \\ 4 \\ -6 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \\ 0 \end{bmatrix}.$$

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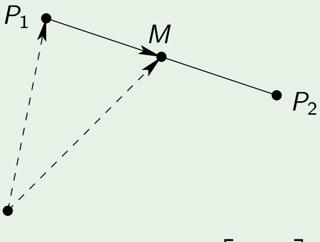

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Therefore $M = M(2,-2,0)$.

Example (§4.1 Exercise 15)

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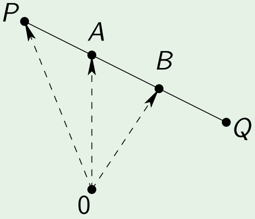

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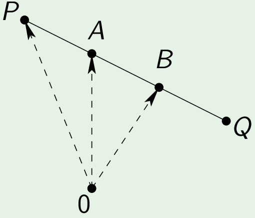

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$\overrightarrow{0A} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$, and $\overrightarrow{0B} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$.

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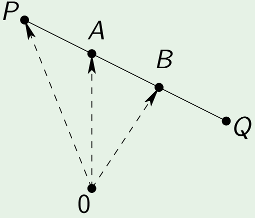


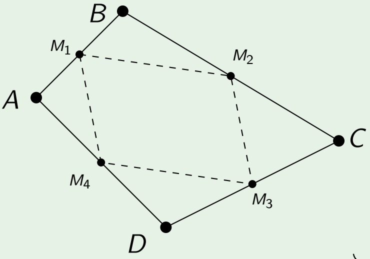
$\overrightarrow{0A} = \overrightarrow{0P} + \frac{1}{3}\overrightarrow{PQ}$ and $\overrightarrow{0B} = \overrightarrow{0P} + \frac{2}{3}\overrightarrow{PQ}$. Since $\overrightarrow{PQ} = \begin{bmatrix} 6 \\ -9 \\ -3 \end{bmatrix}$,

$\overrightarrow{0A} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}$, and $\overrightarrow{0B} = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix} + \begin{bmatrix} 4 \\ -6 \\ -2 \end{bmatrix} = \begin{bmatrix} 6 \\ -3 \\ 3 \end{bmatrix}$.

Therefore, the two points are $A(4,0,4)$ and $B(6,-3,3)$.

Example (§4.1 Example 6)

Let $ABCD$ be an arbitrary quadrilateral. Show that the midpoints of the four sides of $ABCD$ are the vertices of a parallelogram.



Let M_1 denote the midpoint of $\overrightarrow{AB}$,
M_2 the midpoint of $\overrightarrow{BC}$,
M_3 the midpoint of $\overrightarrow{CD}$, and
M_4 the midpoint of $\overrightarrow{DA}$.

It suffices to prove that $\overrightarrow{M_1M_2} = \overrightarrow{M_4M_3}$.

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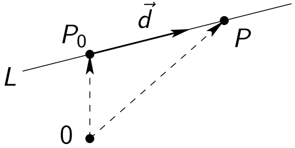
Read §4.1 Example 7 yourselves – determining whether or not two vectors are parallel.

Equations of Lines

Let L be a line, $P_0(x_0, y_0, z_0)$ a fixed point on L, $P(x, y, z)$ an arbitrary point on L, and $\vec{d} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ a direction vector for L, i.e., a vector parallel to L.



Then $\overrightarrow{0P} = \overrightarrow{0P_0} + \overrightarrow{P_0P}$, and $\overrightarrow{P_0P}$ is parallel to $\vec{d}$, so $\overrightarrow{P_0P} = t\vec{d}$ for some $t \in \mathbb{R}$.

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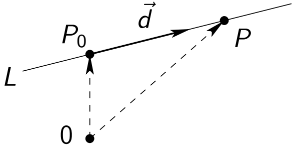

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Vector Equation of a Line

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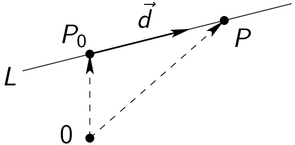

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Vector Equation of a Line

$$\overrightarrow{0P} = \overrightarrow{0P_0} + t\vec{d}, t \in \mathbb{R}.$$

Notation in the text: $\vec{p} = \overrightarrow{0P}$, $\vec{p}_0 = \overrightarrow{0P_0}$, so $\vec{p} = \vec{p}_0 + t\vec{d}$.

In component form, this is written as

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix} + t \begin{bmatrix} a \\ b \\ c \end{bmatrix}, t \in \mathbb{R}.$$

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Parametric Equations of a Line

$$\begin{array}{rcl} x & = & x_0 + ta \\ y & = & y_0 + tb \\ z & = & z_0 + tc \end{array}, \quad t \in \mathbb{R}.$$

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§4.1 Example 11 describes what happens with the parametric equations of a line in $\mathbb{R}^2$.

Example (similar to §4.1 Example 8)

Find an equation for the line through two points $P(2, -1, 7)$ and $Q(-3, 4, 5)$.

Example (similar to §4.1 Example 8)

Find an equation for the line through two points $P(2, -1, 7)$ and $Q(-3, 4, 5)$.

A direction vector for this line is

$$\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

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$$\vec{d} = \overrightarrow{PQ} = \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

Therefore, a vector equation of this line is

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ 7 \end{bmatrix} + t \begin{bmatrix} -5 \\ 5 \\ -2 \end{bmatrix}.$$

Example (similar to §4.1 Example 9)

Find an equation for the line through $Q(4, -7, 1)$ and parallel to the line

$$L : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}.$$

Example (similar to §4.1 Example 9)

Find an equation for the line through $Q(4, -7, 1)$ and parallel to the line

$$L : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}.$$

The line has equation

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ -7 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ -5 \\ 3 \end{bmatrix}, t \in \mathbb{R}.$$

Example

Given two lines L_1 and L_2, find the point of intersection, if it exists.

$$L_1: \begin{array}{rcl} x &=& 3+t \\ y &=& 1-2t \\ z &=& 3+3t \end{array} \qquad\qquad L_2: \begin{array}{rcl} x &=& 4+2s \\ y &=& 6+3s \\ z &=& 1+s \end{array}$$

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Lines L_1 and L_2 intersect if and only if there are values $s, t \in \mathbb{R}$ such that

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i.e., if and only if the system

$$\begin{array}{rcl} 2s-t & = & -1 \\ 3s+2t & = & -5 \\ s-3t & = & 2 \end{array}$$

is consistent.

Example (continued)

$$\left[\begin{array}{cc|c} 2 & -1 & -1 \\ 3 & 2 & -5 \\ 1 & -3 & 2 \end{array}\right] \rightarrow \cdots \rightarrow \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{array}\right]$$

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L_1 and L_2 intersect when $s = -1$ and $t = -1$.

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Using the equation for L_1 and setting $t = -1$, the point of intersection is

$$P(3 + (-1), 1 - 2(-1), 3 + 3(-1)) = P(2, 3, 0).$$

Example (continued)

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Note. You can check your work by setting $s = -1$ in the equation for L_2.

Example (§4.1 Exercise 22(g))

Find equation**s** for the lines through $P(1,0,1)$ that meet the line

$$L : \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

at points distance three from $P_0(1,2,0)$.

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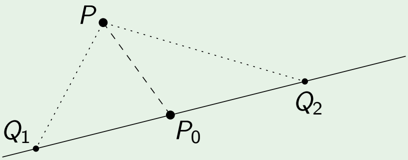
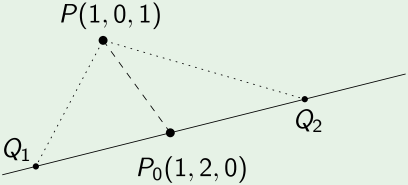


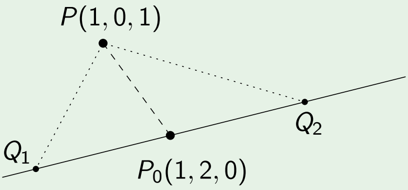
Find points Q_1 and Q_2 on L that are distance three from P_0, and then find equations for the lines through P and Q_1, and through P and Q_2.

Example (continued)



$$\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

Example (continued)



$$\vec{d} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$$

First, $||\vec{d}|| = \sqrt{2^2 + (-1)^2 + 2^2} = \sqrt{9} = 3$, so

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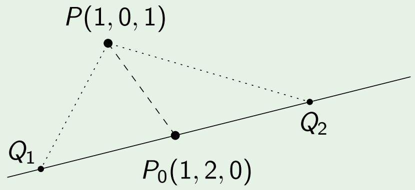
Example (continued)



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$$\overrightarrow{0Q_1} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \text{ and } \overrightarrow{0Q_2} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix},$$

so $Q_1 = Q_1(3, 1, 2)$ and $Q_2 = Q_2(-1, 3, -2)$.

Example (continued)

Equations for the lines:

- the line through $P(1,0,1)$ and $Q_1(3,1,2)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{0P} + \overrightarrow{PQ}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$$

- the line through $P(1,0,1)$ and $Q_2(-1,3,-2)$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \overrightarrow{0P} + \overrightarrow{PQ}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} -2 \\ 3 \\ -3 \end{bmatrix}.$$