Midterm 1 Review (Math 211)
Tips and Tricks

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Summary

Studying

- Work through the practice tests to identify which kinds of problems you need more practice with, and to check your pace.
- Use Lyryx and the exercises in the textbook for extra practice.
- Read the sections of the textbook that you feel shaky on.
- Work in groups.

Gaussian elimination

Some problems will focus on the **process** of reducing a matrix to (reduced) row echelon form. If you're asked about the *rank* of a system, *how many solutions/parameters* a system has, *finding particular solutions*, *writing down all solutions in parametric form*, or *inverting a* 3×3 *matrix* you need to use Gaussian elimination.

1. Convert from the linear system to an augmented matrix.
2. Use row operations to reach row echelon form. Tip: swapping rows or subtracting rows to get a 1 can avoid fractions

You can practice with the applet at `http://people.ucalgary.ca/~roed/courses/211/practice.html`.

Reading off the answer

Suppose you've reduced to (reduced) row-echelon form

- The rank is the number of **nonzero rows**.
- The parameters correspond to **columns with no leading 1**.
- A system is inconsistent (has no solution) if and only if it has a row representing $0 = 1$.
- To solve a system in reduced row echelon form, change it back to equations, solve each for the **variable corresponding to the leading one**.
- To find a particular solution you can **set all parameters to zero**.
- A system will have a unique solution if and only if it has **no parameters** and is **consistent**.

Tip: To find the rank or number of solutions/parameters, you don't need to go all the way to reduced row echelon form.

Gaussian elimination with unknowns

Row reduction becomes a bit tricker if there are unknowns in your matrix, but **the steps are the same**. For example, $\left[\begin{array}{ccc|c} a & 2 & 3 & 1 \\ 1 & 1 & 3 & 4 \end{array}\right]$.

1. Delay dividing by an unknown as long as possible. This means swapping rows to move them to the bottom
2. When you do divide by an expression involving an unknown (e.g. $1 - 2a$), Break the problem into cases depending on whether $1 - 2a$ is zero or not.
3. Each case can end up with a different shape, with different qualitative behavior.

Gaussian elimination with unknowns

$$\left[\begin{array}{ccc|c} a & 2 & 3 & -1 \\ 1 & 1 & 3 & 4 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ a & 2 & 3 & -1 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 2-a & 3-3a & -1-4a \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 2-a & 3-3a & -1-4a \end{array}\right]$$

Case 1: $a = 2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 0 & -3 & -9 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 1 & 0 & -5 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Case 2: $a \neq 2$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 3 & 4 \\ 0 & 1 & \frac{3-3a}{2-a} & \frac{-1-4a}{2-a} \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 3-\frac{3-3a}{2-a} & 4-\frac{-1-4a}{2-a} \\ 0 & 1 & \frac{3-3a}{2-a} & \frac{-1-4a}{2-a} \end{array}\right]$$

Parametric Form from Row Reduced Matrix

1. Switch from matrices back to equations.
2. Solve each equation **for the variable corresponding to the leading 1** in terms of the **parameters**.
3. If needed, rewrite using vectors.

Parametric Form from Row Reduced Matrix

$$\left[\begin{array}{ccccc|c} 1 & 0 & 2 & 0 & 1 & 3 \\ 0 & 1 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Columns 3 and 5 correspond to parameters: s and t.

$$\begin{aligned} & & x_3 &= s \\ & & x_5 &= t \\ x_1 + 2x_3 + x_5 = 3 &\Rightarrow & x_1 &= 3 - 2s - t \\ x_2 + x_3 = 4 &\Rightarrow & x_2 &= 4 - s \\ x_4 - x_5 = 1 &\Rightarrow & x_4 &= 1 + t \end{aligned}$$

To Vectors

$$
\begin{aligned}
x_1 &= 3 - 2s - t \\
x_2 &= 4 - s \\
x_3 &= s \\
x_4 &= 1 + t \\
x_5 &= t
\end{aligned}
$$

$$
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}
$$

Transpose

If A is an $m \times n$ matrix then A^T is $n \times m$, the reflection across the diagonal.

- $(kA)^T = kA^T$
- $(A^T)^T = A$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- A is symmetric if $A = A^T$

Inverse

- $(kA)^{-1} = \frac{1}{k}A^{-1}$
- $(A^{-1})^{-1} = A$
- $(A+B)^{-1} \neq A^{-1} + B^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Solving equations with inverses or transposes

Tip: Transpose undoes transpose and inverse undoes inverse

$$\left(A^{-1} + 3\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\right)^{-1} = 7A$$

$$A^{-1} + 3\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} = \frac{1}{7}A^{-1}$$

$$\frac{6}{7}A^{-1} = -3\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^{-1} = -\frac{7}{2}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A = -\frac{2}{7}\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}^{-1}$$

$$A = -\frac{2}{7}\begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}$$

Conditions for invertibility

The following conditions on a square matrix are all equivalent to invertibility:

- The system $A\mathbf{x} = \mathbf{0}$ has only the trivial solution $\mathbf{x} = \mathbf{0}$,
- A can be transformed to the identity by row operations,
- $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$,
- There is a matrix C with $AC = I$.
- The determinant of A is nonzero (we only know how to compute $\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc$)

Tip: To check that two square matrices A and B are inverses of *each other* we just need to know that $AB = I$.

Finding inverses

For 3×3 or larger inverses, use the inversion algorithm:

1. Write down $\left[A \mid I\right]$.
2. Row reduce. If you can't make the left side into the identity (row of zeros to the centerline), A **is not invertible**.
3. Otherwise, read off the inverse as $\left[I \mid A^{-1}\right]$.

Matrix multiplication

Two methods for matrix-matrix multiplication

1. Use the dot product to compute the (i, j)-entry of AB: the i-th row of A dotted with the the j-th column of B.
2. Break B up into columns $\mathbf{b}_1, \ldots, \mathbf{b}_k$ and determine the columns of the product as $A\mathbf{b}_1, \ldots, A\mathbf{b}_k$

Warning: $AB \neq BA$ in general

Two methods for matrix-vector multiplication

1. Use the dot product, as for matrix-matrix multiplication
2. If $\mathbf{b} = \begin{bmatrix} b_1 & \cdots & b_n \end{bmatrix}^T$ and the columns of A are $\mathbf{a}_1, \ldots, \mathbf{a}_n$, then

$$A\mathbf{b} = \mathbf{a}_1 b_1 + \cdots + \mathbf{a}_n b_n$$

Note: the result $A\mathbf{b}$ is a **linear combination** of the columns of A

Solving equations

Tip: Use your intuition for normal equations to help with matrix equations. Ask yourself what you would do if it were an equation involving numbers

The main differences are:

- You need to be careful about order of multiplication and sizes of products
- There are more situations where you can't divide (in which case you generally use Gaussian elimination)

For example, if we need

$$AYB = C$$

we should "divide" by A and B. Doing it on the correct side gives

$$Y = A^{-1}CB^{-1}.$$

Transformations

- Rotation: $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$
- Expansion / compression: $\begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
- x-expansion / compression: $\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$
- y-expansion / compression: $\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$
- x-shear: $\begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}$
- y-shear: $\begin{bmatrix} 1 & 0 \\ -4 & 1 \end{bmatrix}$
- Reflection: $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

General Suggestions

- Show up at least 5 minutes early: we'll be starting at 7pm.
- Check your work. It's easy to make arithmetic errors.
- Pace yourself. If you're taking a long time on a problem, come back to it later.
- Get enough sleep on Monday night.