18.02 Problem Set 2

(Due Thursday, February 17, 11:59:59 PM)

Part I (40 points)

HAND IN ONLY THE UNDERLINED PROBLEMS
(The others are *some* suggested choices for more practice.)
EP = Edwards and Penny; SN = Supplementary Notes (most have solutions)

Solving linear systems, equations of planes
Reading: EP §12.4, SN §M
Exercises:
EP §12.4 6, <u>13</u>, 17, <u>32</u>, 48, <u>54</u>, 60
SN §1G 1
SN §1H <u>6</u>, 7, 8

Parametric equations for lines, curves and surfaces
Reading: EP §§10.4, 12.4, 12.5
Exercises:
EP §10.4 30, <u>41</u>, <u>42</u>
SN §1I 3<u>cd</u>, 5, <u>7</u>

Derivatives of vector functions
Reading: EP §12.5
Exercises:
EP §12.5 7, 10, 24, <u>39</u>, <u>55</u>, 62
SN §1J 5, 8

Part II (60 points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently.

Problem 1 (10; 2, 2, 3, 3)
Cookies, doughnuts and croissants contain essentially the same ingredients (flour, sugar, egg, butter), but in different proportions. It takes 22 grams of flour, 18 grams of sugar, 5 grams of egg and 10 grams of butter to make a cookie. A doughnut, on the other hand, requires 40 grams of flour, 10 grams of sugar, 14 grams of egg and 10 grams of butter. Finally, a croissant takes 50 grams of flour, 3 grams of sugar,

5 grams of egg and 22 grams of butter. Consider an assortment of x_1 cookies, x_2 doughnuts and x_3 croissants, and form the column vector

$$\mathbf{X} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}.$$

a) Write a matrix equation relating $\mathbf{X}$ to the amount of flour, sugar, egg and butter used to make $\mathbf{X}$.
b) Each of the four ingredients has a specific nutritional value. For example, 100 g of flour contains 10 g of protein, 76 g of carbohydrates, and 1 g of fat. Sugar consists purely of carbohydrates, while 1 g of butter contains 0.82 grams of fat. Eggs are a mixture: 13% protein, 1% carbohydrates and 10% fat. Write another matrix equation that gives the total nutritional value of the assortment of pastries.
c) Give a matrix formula expressing the numbers x_i of pastries of each type which will add up to y_1 g of protein, y_2 g of carbohydrates and y_3 g of fat. Express your answer in the form $\mathbf{X} = A\mathbf{Y}$ (you may use a calculator or computer to find A).
d) The recommended daily amounts of protein, carbohydrates and fat for a 2000 calorie diet are 50, 300 and 65 grams respectively. If you wanted to follow these guidelines while eating only cookies, doughnuts and croissants, how many pastries of each type should you eat daily?

Problem 2 (9; 2, 4, 3)
a) List the three possible types of sets that can occur as the intersection of a line and a plane in $\mathbb{R}^3$.
b) Consider the system of equations $\begin{cases} 2x_1 - x_2 + x_3 = 0 \\ x_1 + x_2 + x_3 = 0 \\ c_1x_1 + c_2x_2 + c_3x_3 = b \end{cases}$
Find a vector in the direction of the line of intersection of the planes represented by the first and second equations. List all possibilities from (a) that can occur as the intersection of this line with the third plane when $b = 0$. Find an algebraic condition on $\mathbf{c} = (c_1, c_2, c_3)$ (in the simplest form you can) that distinguishes between the possibilities. Also express in words (geometrically) what your conditions say about the three vectors $\mathbf{u} = (2, -1, 1), \mathbf{v} = (1, 1, 1)$, and $\mathbf{c}$.
c) Do the same as in part (b) for $b \neq 0$.

Problem 3 (7; 2, 1, 2, 2)
To represent 3-dimensional objects on a computer screen, one needs to draw a given point $P = (x_1, x_2, x_3)$ at the place where the line from P to the eye meets the screen. Suppose that the screen is the yz-plane, and the eye is at $E = (r, 0, 0)$ (assume $r > 0$).
a) At what point $Q = (y, z)$ in the yz-plane should one represent the point P? (Assume that $x_1 < r$. Why can we make this assumption?)
b) What does the image on the screen of a line segment in space look like? Why?

c) Vanishing points are points on the two-dimensional image where parallel lines meet at infinity. Consider a vector $\mathbf{A} = (a_1, a_2, a_3)$ with $a_1 < 0$, and consider the line given by parametric equations $x_i = ta_i$, parallel to $\mathbf{A}$ through the origin. Compute the limiting value of (y, z) as $t \to \infty$. Write a parametrization of the line parallel to $\mathbf{A}$ through another point $R = (r_1, r_2, r_3)$, and find the limiting value of (y, z) for this line. Why did we omit the cases $a_1 = 0$ and $a_1 > 0$?

d) Consider the two directions $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ with $a_1 < 0$ and $b_1 < 0$, and assume that $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. Let $\mathbf{A}$ and $\mathbf{B}$ be the vectors from the origin $(0, 0)$ to the vanishing points corresponding to $\mathbf{a}$ and $\mathbf{b}$. Show that the dot product $\mathbf{A} \cdot \mathbf{B}$ is the same no matter what $\mathbf{a}$ and $\mathbf{b}$ are.

Problem 4 (25; 3, 3, 4, 6, 6, 3)

Parametrize each of the following curves or surfaces:

a) The ellipse lying in the plane $x + 3y + 4z = 0$, centered at the origin, with long axis of length 4 in the xy-plane and a perpendicular short axis of length 1.

b) The cone with vertex half-angle α and axis pointing along the positive x-axis.

c) A circular torus or inner radius 3 and outer radius 5.

d) A Mobius strip (a Mobius strip can be formed by taking a rectangular strip of paper, making a half twist and then taping the ends together). You may choose any constants necessary, or leave them as variables.

e) A spiraling seashell: a tubular surface centered on the helix $(\cos t, \sin t, t)$, with radius away from the helix proportional to t.

f) Make up your own curve or surface and parametrize it. Include a drawing or printout of your curve or surface.

Problem 5 (9; 4, 3, 2)

a) Consider the parametric curve given by

$$\gamma(t) = (4 \cos t \; e^{2t}, 2 \sin t \; e^{t} + t^2, t^3 - t^2).$$

As $\gamma(t)$ is traced out, the tangent vector traces out another curve, $\beta(t)$ (where here we consider the base of the tangent vector at time t to be attached to the curve $\gamma(t)$). Find a parametric equation for $\beta(t)$.

b) Apply the same process to $\beta(t)$, yielding $\alpha(t)$.

c) Write down an expression for $\alpha(t)$ involving only $\gamma(t)$.