

18.02 ESG Final Exam

Spring 2005

Write your name in the top right corner of this page. Work in the space provided or on the backs of pages. You are allowed five pages of notes and the use of at most two calculators, but you must show your work to get full credit and no other aids are allowed.

There are 255 points available. Full credit is 250.

1. [20 points]
Consider the system of equations

$$\begin{aligned}x_1 + 5x_2 + 3x_3 &= 1 \\x_1 + 7x_2 + 7x_3 &= 7 \\3x_1 + 19x_2 + 17x_3 &= -2\end{aligned}$$

- (a) [3] Rewrite this system in the matrix form $A\mathbf{x} = \mathbf{b}$.
- (b) [5] Is A invertible? Justify your answer.
- (c) [12] Give all solutions to the system. Make sure to include your steps.

2. [10 points]

Consider the map $f: \mathbb{R}^5 \rightarrow \mathbb{R}^2$ given by the formula

$$f(x, y, z, w, u) = (5x + 2y - 3z + u, u - z + 4w).$$

What is the dimension of the kernel of f ?

3. [15 points]

Let $\mathbf{v} = (7, 6, 5)$ and $\mathbf{w} = (3, 2, -1)$. Express \mathbf{v} as the sum of two perpendicular vectors, one of which points in the direction of \mathbf{w} . [Hint: project \mathbf{v} onto \mathbf{w}]

4. [25 points]

Find all critical points of the function $f(x, y, z) = xy^3 - z$ when restricted to the surface $xy + yz = -3$

5. [20 points]

Consider the function $f: \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined by

$$f(x, y, z) = (x^2 y - 2 x z + 1, 5 z^3 + 4 x y^2).$$

Approximate f by a linear function near the point $(1, 1, 0)$.

6. [25 points]

Let $f(x, y) = 3xy - x^3 - y^3$. Find and classify all critical points of f .

7. [20 points]

Find the area of the ellipse $(4x-y)^2 + (x-3y)^2 < 1$ using an appropriate change of coordinates.

8. [25 points]

Let $\mathbf{F}(x, y, z) = (ax^2y + z^2, x^3 + 4y^3z, bxz + y^4)$.

- (a) [6] For what values of a and b will \mathbf{F} be conservative?
- (b) [12] Using these values of a and b , find a function $f(x, y, z)$ such that $\mathbf{F} = \nabla f$.
- (c) [7] Again using these values of a and b , give a defining equation for a surface S with the property that

$$\int_P^Q \mathbf{F} \cdot \mathbf{T} \, ds = 0$$

for any two points P and Q lying on S and any path between them.

9. [15 points]

Let C be the spiral given in polar coordinates by the equation $r = \theta$, $0 \leq \theta \leq 2\pi$, traced out starting from the origin. Let $\mathbf{F}(x, y) = (x^2, y)$. Reduce the following line integral to a single variable integral, but do not evaluate the resulting single variable integral:

$$\int_C \mathbf{F} \cdot d\mathbf{r}.$$

10. [55 points]

Consider the sphere S of radius 1 centered at the point $(0, 0, 1)$

(a) [10] Parameterize S . [Hint: how would the parameterizations of a sphere centered at the origin and a sphere centered at $(0, 0, 1)$ differ?]

(b) [15] Let $f(x, y, z) = x^2 z + y^2 z - x^2 - y^2$. Compute

$$\iint_S f(x, y, z) dS.$$

[Can you tell what the scaling factor (ie $|\mathbf{r}_u \times \mathbf{r}_v|$) is at a glance?]

(c) [15] Let $\mathbf{F}(x, y, z) = (x + 3y, 2y - z, 4z + x)$. Compute

$$\oiint_S \mathbf{F} \cdot \mathbf{n} dS.$$

(d) [15] Now consider just the upper half of S (above the $z = 1$ plane). Call this surface S_1 and equip it with the upward pointing normal. Let $\mathbf{F} = (z - y - 1, x + z^2 + z - 2, xy^2 + xz - 4)$. Compute

$$\iint_{S_1} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

11. [25 points] Consider the part of the surface $z = -r^2 + 3r - 2$ that lies above the xy -plane (r is the r of cylindrical coordinates). Call this surface S . Let

$$\mathbf{F}(x, y, z) = (x + 3y - \sin^4(z^5), x^2 - y^2, -z - 4).$$

Evaluate

$$\iint_S \mathbf{F} \cdot \mathbf{n} \, dS.$$

[Hint: use the divergence theorem so that the surface integral that you actually compute is easier.]