$\qquad$

### 18.02 $\rho$ Exam 2

Work in the space provided (you may use the back). You may use a calculator and one page of notes. Circle your answer!

1) (50 pts.) Consider the surface $z=f(x, y)=\left(2 x^{2}+3 y^{2}\right) e^{-x^{2}-y^{2}}$.
(a) (5 points) Find the first and second order partials of $f$
(b) (10 points) Now imagine pouring water onto this surface at the point $\left(\frac{1}{2}, \frac{1}{3}\right)$. What is the tangent plane to the surface at this point?
(c) (5 points) In what direction will the water flow?
(d) (5 points) At what angle downward will the water flow? Draw a picture to show me how you're defining your angle.
(e) (20 points) As the water level rises, it will gradually fill up a bounded $x y$ region until spilling over and flowing out to infinity. At what depth will this occur and at what point (or points) will the water spill over? Justify your answer using techniques from 18.02 (rather than the simplicity of the function $f$ ).
(f) (10 points) Now imagine pouring the water at an arbitrary point $\left(x_{0}, y_{0}\right)$. The water will flow downhill. For what region of initial points will the water flow inward? (ie more toward the origin than away?) You may specify this region by giving an inequality.
2) (20 pts.) Suppose $f: \mathbb{R}^{2} \rightarrow \mathbb{R}, g: \mathbb{R}^{2} \rightarrow \mathbb{R}, h: \mathbb{R}^{2} \rightarrow \mathbb{R}, x: \mathbb{R} \rightarrow \mathbb{R}$, and $y: \mathbb{R} \rightarrow \mathbb{R}$. Suppose that you know that $x(\pi)=2$, that $y(\pi)=4$, that $g(2,4)=-1$ and $h(2,4)=0$. In addition, I tell you that $x^{\prime}(\pi)=1$, that $y^{\prime}(\pi)=-1$, that the gradient of $g$ at $(1,-1)$ is $(8,0)$, that the gradient of $g$ at $(2,4)$ is $(5,3)$, that $h(x, y)=x^{3}-x y$, and that the gradient of $f$ at $(-1,0)$ is $(2,3)$. Let $\alpha(t)=f(g(x(t), y(t)), h(x(t), y(t)))$. Find $a^{\prime}(\pi)$.
3) ( 20 pts.) Show that the sphere $x^{2}+y^{2}+z^{2}=r^{2}$ and the cone $z^{2}=\frac{x^{2}}{a^{2}}+\frac{y^{2}}{b^{2}}$ are orthogonal (that is, have perpendicular tangent planes) at every point of their intersection.
4) (10 pts.) Suppose you are trying to maximize the function $f$ subject to the constraint $g(x, y)=0$. I have drawn the graph of $g$ below, as well as a selection of level curves of $f$. Assuming that the level curves of $f$ vary smoothly in between those I've drawn, circle the points on the graph of $g$ that could be maxima of $f$.

## Extra Credit

5) (5 pts.) In problem 1, find the region of initial points for which the water will pool initially rather than flow off toward infinity.
