18.02 ESG Exam 4 Solutions
Spring 2005

1. [30 points]
Consider the ellipsoid E given by the equation $\frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{36} = 1$. Let $f(x, y, z) = \sqrt{2x^2 + 2y^2 + 1}$. Calculate the value of

$$\iint_E f(x, y, z)dS$$

[Hint: parameterize E similarly to a sphere.]

Solution:
We parameterize E by

$$\begin{aligned} x &= 2\cos\theta\sin\phi \\ y &= 2\sin\theta\sin\phi \\ z &= 6\cos\phi \\ &0 \le \theta \le 2\pi, 0 \le \phi \le \pi \end{aligned}$$

We now compute the area element for the surface integral:

$$\begin{aligned} \left|\frac{\partial\vec{r}}{\partial\theta} \times \frac{\partial\vec{r}}{\partial\phi}\right| &= |(-2\sin\theta\sin\phi, 2\cos\theta\sin\phi, 0) \times (2\cos\theta\cos\phi, 2\sin\theta\cos\phi, -6\sin\phi)| \\ &= \left|(-12\cos\theta\sin^2\phi, -12\sin\theta\sin^2\phi, -4\sin\phi\cos\phi)\right| \\ &= \sqrt{144\sin^4\phi + 16\sin^2\phi\cos^2\phi} \\ &= 4\sin\phi\sqrt{9\sin^2\phi + \cos^2\phi} \\ &= 4\sin\phi\sqrt{8\sin^2\phi + 1} \end{aligned}$$

Thus the integral is

$$\begin{aligned} \iint_E f(x, y, z)dS &= \int_0^{2\pi}\int_0^{\pi} \sqrt{2x^2 + 2y^2 + 1} \cdot 4\sin\phi\sqrt{8\sin^2\phi + 1}\, d\phi\, d\theta \\ &= 2\pi\int_0^{\pi} 4\sin\phi(8\sin^2\phi + 1)\, d\phi \\ &= \frac{304\pi}{3} \end{aligned}$$

2. [30 points] Let S be the part of the paraboloid $z = 1-x^2-y^2$ above the xy-plane, and let $\mathbf{F}(x, y, z) = (x^2+yz^4-\sin(z^2), x-4y+z, 4z+1)$. Compute

$$\iint_S \mathbf{F} \cdot \mathbf{n} dS$$

by using the divergence theorem to reduce to a simpler surface integral.

Solution:
We want to use the simpler disc of radius 1 centered at the origin in the complex plane. Let T be this surface, with upward pointing normal. Let V be the volume bounded by S and T. By the divergence theorem,

$$\iiint_V \nabla \cdot \mathbf{F}\, dV = \iint_S \mathbf{F} \cdot \mathbf{n} dS - \iint_T \mathbf{F} \cdot \mathbf{n} dS.$$

We first compute the divergence:

$$\begin{aligned} \nabla \cdot \mathbf{F} &= \frac{\partial}{\partial x}(x^2 + yz^4 - \sin(z^2)) + \frac{\partial}{\partial y}(x - 4y + z) + \frac{\partial}{\partial z}(4z + 1) \\ &= 2x - 4 + 4 \\ &= 2x. \end{aligned}$$

Since V is symmetric with respect to the yz plane, $\iiint_V 2x\, dV = 0$. Thus, $\iint_S \mathbf{F} \cdot \mathbf{n} dS = \iint_T \mathbf{F} \cdot \mathbf{n} dS$. So

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n} dS &= \int_0^{2\pi} \int_0^1 (x^2, x - 4y, 1) \cdot (0, 0, 1) r dr\, d\theta \\ &= \pi. \end{aligned}$$

3. [40 points]
Consider the torus of inner radius 3, outer radius 5, central axis the y-axis and with central plane the xz-plane. Let S be the part of this torus above the xy-plane, with outward pointing normal. Define $\mathbf{F}(x, y, z) = (xy + \cos(\frac{\pi}{2} e^{xyz}) \sin(xy), x + \ln(x^2 + y^2 + z^2) z^{x^2+y^2}, x^2 + xy^4 - 2x\cos(y) - e^{5y-3\cos(x)})$. Compute

$$\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

Solution: The boundary of S is the disjoint union of two circles in the xy-plane, both oriented counterclockwise when viewed from above.

Call the discs enclosed within the circles D_1 and D_2, oriented upward. We use Stokes' Theorem and then Green's theorem, noting that on the two discs, $z = 0$.

$$\begin{aligned}\iint_S (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS &= \iint_{D_1 \cup D_2} \frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(xy)\, dA \\ &= \iint_{D_1 \cup D_2} 1 - x\, dA \\ &= 2\pi.\end{aligned}$$

where this last step uses the fact that $D_1 \cup D_2$ is symmetric about the y-axis.