18.02 ESG Exam 3 Solutions
Spring 2005

1. [30+20EC points]
Set up the integration in each part, but you do not need to evaluate the integral. You may use rectangular, cylindrical or spherical coordinates as you wish.

(a) [15] Consider the region bounded by the two cylinders $y^2 + z^2 = 1$ and $x^2 + z^2 = 1$. Find the average distance of a point in this region to the point on both cylinders with largest z-coordinate.

Solution: The problem certainly doesn't have spherical symmetry, so spherical coordinates seems like a poor choice. I've included the argument for cartesian below and the answer for cylindrical as well.

The point on both centers with the largest z coordinate is $(0, 0, 1)$ and the distance from an arbitrary point (x, y, z) to $(0, 0, 1)$ is $\sqrt{x^2 + y^2 + (z-1)^2}$. We can split the region up into eight identical vertical sections, and the average value of the distance over the whole region will be the same as the average value over any one. So the answer in cartesian is:

$$\frac{\int_0^1 \int_0^x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{x^2 + y^2 + (z-1)^2}\, dz\, dy\, dx}{\int_0^1 \int_0^x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} 1\, dz\, dy\, dx}.$$

In cylindrical,

$$\frac{\int_0^{\frac{\pi}{4}} \int_0^{\sec(\theta)} \int_{-\sqrt{1-(r\cos(\theta))^2}}^{\sqrt{1-(r\cos(\theta))^2}} \sqrt{r^2 + (z-1)^2}\, r\, dz\, dr\, d\theta}{\int_0^{\frac{\pi}{4}} \int_0^{\sec(\theta)} \int_{-\sqrt{1-(r\cos(\theta))^2}}^{\sqrt{1-(r\cos(\theta))^2}} r\, dz\, dr\, d\theta}.$$

(b) [15] Consider a solid circular torus of inner radius 3 and outer radius 5. If the density of the torus is given by the square of the distance to the center, find the mass of the torus.

Solution: Since the torus has cylindrical symmetry, we'll use cylindrical coordinates. Then we're integrating the function $r^2 + z^2$, so the mass of the torus is

$$2\int_0^{2\pi}\int_3^5\int_0^{\sqrt{1-(r-4)^2}}(r^2+z^2)r\,dz\,dr\,d\theta.$$

(c) [20 extra credit] Find the average distance between two points in the unit disk.

Solution: We integrate as two points (r, θ) and (s, φ) run over a pair of disks (actually over the direct product of two disks sitting in $\mathbb{R}^4$). The area of this region is π^2 (which you can find by integrating the function 1), so the average distance between the two points is

$$\frac{1}{\pi^2}\int_0^{2\pi}\int_0^1\int_0^{2\pi}\int_0^1\sqrt{r^2+s^2-2rs\cos(\theta-\varphi)}rs\,ds\,d\varphi\,dr\,d\theta.$$

2. [20 points] In this problem, set up the integration AND evaluate. Integrate the function $x^2 + z^2$ over the region enclosed by the planes $z = 0$ and $z = 1$ and the cone $z^2 = x^2 + y^2$.

Solution: Cylindrical seems reasonable since we have cylindrical symmetry.

$$\begin{aligned}
\iiint_R f(x,y,z)dV &= \int_0^{2\pi}\int_0^1\int_r^1(r^2\cos(\theta)^2+z^2)r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1(\cos(\theta)^2(r^3-r^4)+\frac{1}{3}(r-r^4))\,dr\,d\theta \\
&= \int_0^{2\pi}(\frac{1}{20}\cos(\theta)^2+\frac{1}{10})\,dr\,d\theta \\
&= \frac{\pi}{20}+\frac{2\pi}{10} \\
&= \frac{\pi}{4}
\end{aligned}$$

3. [25 points]
For each of the following vector fields, determine whether it is conservative. If it is, find a potential function.

(a) $\vec{F_1}(x, y, z) = (4x^3y + 3z, x^4 - z + 2, 3x + y)$.
Solution: Three dimensional vector fields are conservative iff their curl is zero.

$$\begin{aligned}\nabla \times \vec{F_1} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4x^3y + 3z & x^4 - z + 2 & 3x + y \end{pmatrix} \\ &= (1 - (-1), 3 - 3, 4x^3 - 4x^3) \\ &\neq (0, 0, 0),\end{aligned}$$

and thus $\vec{F_1}$ is not conservative.

(b) $\vec{F_2}(x, y, z) = (2x + y\cos(xy), x\cos(xy) - ze^{yz}, 2 - ye^{yz})$.
Solution: Same thing:

$$\begin{aligned}\nabla \times \vec{F_2} &= \det \begin{pmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x + y\cos(xy) & x\cos(xy) - ze^{yz} & 2 - ye^{yz} \end{pmatrix} \\ &= (-e^{yz} - yze^{yz} + e^{yz} + yze^{yz})\mathbf{i} + (0 - 0)\mathbf{j} \\ &\qquad + (\cos(xy) - xy\sin(xy) - \cos(xy) + xy\sin(xy))\mathbf{k} \\ &= (0, 0, 0),\end{aligned}$$

so $\vec{F_2}$ is conservative.
Suppose that $F_2 = \nabla f$. Integrating the first component we see that

$$f(x, y, z) = x^2 + \sin(xy) + g(y, z).$$

Differentiating with respect to y gives

$$x\cos(xy) + \frac{\partial g}{\partial y} = x\cos(xy) - ze^{yz}$$

and thus

$$g(y, z) = -e^{yz} + h(z).$$

Finally, differenting our updated f with respect to z gives

$$-ye^{yz} + \frac{dh}{dz} = 2 - ye^{yz}$$

and thus

$$h(z) = 2z + C.$$

Putting it all together we have that

$$x^2 + \sin(xy) - e^{yz} + 2z$$

is a potential function for $\vec{F_2}$.

4. [25 points] Use any method we've learned to evaluate the following line integrals.

(a) $\int_{C_1} \vec{F_1}(x,y,z) \cdot d\vec{s}$ where $\vec{F_1}(x,y,z) = (4x^3y + 3z, x^4 - z + 2, 3x + y)$ as in $(3a)$ and C_1 is the portion of the parabola $x = y = z^2$ from $(1, 1, -1)$ to $(1, 1, 1)$.

Solution: While $\vec{F_1}$ was not conserative, it was almost conservative. Adding $(0, 2z, 0)$ to $\vec{F_1}$ gives ∇f, where $f(x,y,z) = x^4y + 3xz + yz$ and thus

$$\begin{aligned}
\int_{C_1} \vec{F_1}(x,y,z) \cdot d\vec{s} &= \int_{C_1} (\nabla f - (0, 2z, 0)) \cdot d\vec{s} \\
&= f(1,1,1) - f(1,1,-1) - \int_{-1}^{1} (0, 2t, 0) \cdot (2t, 2t, 1) dt \\
&= 5 - (-3) - \int_{-1}^{1} 4t^2 dt \\
&= \frac{16}{3}.
\end{aligned}$$

(b) $\int_{C_2} \vec{F}_2(x,y,z) \cdot d\vec{s}$ where $\vec{F}_2(x,y,z) = (2x + y\cos(xy), x\cos(xy) - ze^{yz}, 2 - ye^{yz})$ as in $(3b)$ and C_2 is the twisted cubic given by $\gamma(t) = (t, t^2, t^3), 0 \leq t \leq 1$.

Solution: This time our vector field actually is conservative, and all we need to do is plug in the endpoints.

$$\begin{aligned}\int_{C_2} \vec{F}_2(x,y,z) \cdot d\vec{s} &= (1^2 + \sin((1)(1)) - e^{(1)(1)} + 2(1)) \\ &\quad - (0^2 + \sin((0)(0)) - e^{(0)(0)} + 2(0)) \\ &= 2 + \sin(1) - e.\end{aligned}$$