18.02 ESG Exam 2 Solutions
Spring 2005

1. [80 points]
Let $f(x, y) = 4xy - x^3y - xy^3$.

(a) [10] Compute all first and second order partial derivatives of f.

$$\frac{\partial f}{\partial x} = 4y - 3x^2y - y^3.$$
$$\frac{\partial f}{\partial y} = 4x - x^3 - 3xy^2.$$
$$\frac{\partial^2 f}{\partial x^2} = -6xy.$$
$$\frac{\partial^2 f}{\partial x \partial y} = 4 - 3x^2 - 3y^2.$$
$$\frac{\partial^2 f}{\partial y^2} = -6xy.$$

(b) [30] Find and classify (as a local minimum, maximum or saddle point) all critical points of f.

$4y - 3x^2y - y^3 = y(4 - 3x^2 - y^2) = 0$, which implies that $y = 0$ or $4 - 3x^2 - y^2 = 0$.

In the first case, $4x - x^3 = 0$ and thus $x = 0, 2$, or -2.

In the second case, $y^2 = 4 - 3x^2$, so the second equation turns into $4x - x^3 - 3x(4 - 3x^2) = 8x^3 - 8x = 0$ and therefore $x = 0, 1$, or -1.

If $x = 0$ then $y^2 = 4$ so $y = \pm 2$.

If $x = \pm 1$ then $y^2 = 4 - 3 = 1$ so $y = \pm 1$.

Therefore there are nine solutions: $(0, 0)$, $(2, 0)$, $(-2, 0)$, $(0, 2)$, $(0, -2)$, $(1, 1)$, $(1, -1)$, $(-1, 1)$, $(-1, -1)$.

We construct a table to determine what type of critical point it is.

point	$\frac{\partial^2 f}{\partial x^2}$	$\frac{\partial^2 f}{\partial y^2}$	$\frac{\partial^2 f}{\partial x \partial y}$	$\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y}\right)^2$	type of critical point
$(0,0)$	0	0	4	-16	saddle
$(2,0)$	0	0	-8	-64	saddle
$(0,2)$	0	0	-8	-64	saddle
$(-2,0)$	0	0	-8	-64	saddle
$(0,-2)$	0	0	-8	-64	saddle
$(1,1)$	-6	-6	-2	32	maximum
$(1,-1)$	6	6	-2	32	minimum
$(-1,1)$	6	6	-2	32	minimum
$(-1,-1)$	-6	-6	-2	32	maximum

(c) [15] Suppose you now consider f as a function on the disk D of radius 2 around the origin. Write down the Lagrange multiplier equations for finding extrema of f on the circle of radius 2 around the origin. Find the global minimum and global maximum value of f on D.

The boundary of the disk is given by the equation $x^2 + y^2 = 4$. Thus the Lagrange multiplier equations are

$$\begin{aligned} 4y - 3x^2y - y^3 &= 2\lambda x, \\ 4x - x^3 - 3xy^2 &= 2\lambda y, \\ x^2 + y^2 &= 4. \end{aligned}$$

Note that f factors as $xy(4 - x^2 - y^2)$ and is thus identically zero on the circle of radius 2. Thus the maximum of f on the disc occurs at $(1,1)$ and $(-1,-1)$ with value 2 and the minimum at $(1,-1)$ and $(-1,1)$ with value -2.

(d) [5] Find the gradient of f at the point $(2,2)$.

$\nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}) = (4\cdot(2) - 3\cdot(2)^2(2) - (2)^3, 4\cdot(2) - 3\cdot(2)^2(2) - (2)^3) = (-24, -24)$.

(e) [10] Find the tanget plane to f at the point $(2, 2)$.

The equation of the tangent plane is given by

$$z - z_0 = \left.\frac{\partial f}{\partial x}\right|_{(x_0,y_0)} (x - x_0) + \left.\frac{\partial f}{\partial y}\right|_{(x_0,y_0)} (y - y_0)$$

which in this case yields

$$z + 16 = -24(x - 2) - 24(y - 2).$$

(f) [10] Find a direction $\mathbf{u}$ such that $D_{\mathbf{u}}(f) = \frac{-24}{5}$.

We want $\nabla f \cdot \mathbf{u} = \frac{-24}{5}$ so $-24u_x - 24u_y = \frac{-24}{5}$, and since $\mathbf{u}$ is a unit vector, $u_x^2 + u_y^2 = 1$.
Solving these two equations yields $\mathbf{u} = (\frac{4}{5}, -\frac{3}{5})$ or $\mathbf{u} = (-\frac{3}{5}, \frac{4}{5})$.

2. [20 points]
Suppose that the variables x, y, z satisfy an equation $g(x, y, z) = 0$. Assume the point $P(1, 1, 1)$ lies on this level surface of g and that $\nabla g(1, 1, 1) =< -1, 1, 2 >$. Let $f(x, y, z)$ by another function, and assume that $\nabla f(1, 1, 1) =< 1, 2, 1 >$. Find the gradient of the function $w = f(x, y, z(x, y))$ of the two independent variables x and y at the point $x = 1$, $y = 1$.

By the chain rule,

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x}$$
$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$$

and

$$\frac{\partial g}{\partial x} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial x} = 0$$
$$\frac{\partial g}{\partial y} + \frac{\partial g}{\partial z}\frac{\partial z}{\partial y} = 0$$

Thus

$$\frac{\partial w}{\partial x} = \frac{\partial f}{\partial x} - \frac{\partial f}{\partial z}\frac{\frac{\partial g}{\partial x}}{\frac{\partial g}{\partial z}} = 1 - 1 \cdot \frac{-1}{2} = \frac{3}{2}$$

and

$$\frac{\partial w}{\partial y} = \frac{\partial f}{\partial y} - \frac{\partial f}{\partial z}\frac{\frac{\partial g}{\partial y}}{\frac{\partial g}{\partial z}} = 2 - 1 \cdot \frac{1}{2} = \frac{3}{2}.$$

Therefore $\nabla w = (\frac{3}{2}, \frac{3}{2})$.