18.02ρ Problem Set 8

(Due Wednesday, November 23, 13:05:00)

Part I (55 points)

HAND IN ONLY THE UNDERLINED PROBLEMS
(The others are *some* suggested choices for more practice.)
EP = Edwards and Penny; SN = Supplementary Notes (most have solutions)

Surface integrals
Reading: EP §§14.8, 15.5 SN §V9
Exercises:
EP §14.8 <u>9</u>, 12, 14, 16 EP §15.5 <u>3</u>, 5, <u>11</u>, 15, 16, 22, <u>24</u>, <u>37</u> (Don't evaluate the integral)
SN §6B <u>1</u>, 2, 6, <u>7</u>, 12

Green's Theorem
Reading: EP §15.4
Exercises:
EP §15.4 <u>1</u>, 4, <u>9</u>, 18, <u>33</u>, <u>39</u>, 41

Part II (45 points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently. There are more than 45 points available: you may choose which you want to solve in order to obtain enough (though you're welcome to solve more).

Problem 1 (25) In quantum mechanics, the state of a particle can be described by it's wave function Ψ. The wave function is a complex valued function defined on $\mathbb{R}^3$. One interpretation of the wave function is that the probability $P(V)$ of finding the particle in a given volume V of space is

$$P(V) = \iiint_V |\Psi|^2 \, dV.$$

Consider a hydrogen atom. The electron can occupy many different possible orbitals, all of which have different wave functions. For this problem we will be concentrating on the $3p_z$ orbital, which has equation

$$\Psi(\rho, \theta, \phi) = \frac{\sqrt{2}}{81\sqrt{\pi}} \left(\frac{1}{a}\right)^{\frac{3}{2}} (6\sigma - \sigma^2) e^{-\sigma/3} \cos\phi,$$

where $a = \frac{\epsilon_0 h^2}{\pi q^2 m_e} \approx 0.529$Åis the Bohr radius and $\sigma = \frac{\rho}{a}$.

Remark: The general formula for the orbital with quantum numbers n, l and m is

$$\sqrt{\left(\frac{2}{na}\right)^3 \frac{(n-l-1)!}{2n[(n+l)!]^3}} e^{-\frac{\rho}{na}} \left(\frac{2\rho}{na}\right)^l L_{n-l-1}^{2l+1}(\frac{2\rho}{na}) Y_l^m(\phi,\theta)$$

where

$$Y_l^m(\phi,\theta) = \epsilon\sqrt{\frac{2l+1}{4\pi}\frac{(l-|m|)!}{(l+|m|)!}} e^{im\theta} P_l^m(\cos\phi)$$

with $L_q(x) = e^x \frac{d^q}{dx^q}(e^{-x}x^q)$ the qth Laguerre polynomial, $L_{q-p}^p(x) = (-1)^p \frac{d^p L_q}{dx^p}$ an associated Laguerre polynomial, $P_l(x) = \frac{1}{2^l l!}\frac{d^l}{dx^l}(x^2-1)^l$ the $l$the Legendre polynomial, $P_l^m(x) = (1-x^2)^{|m|/2}\frac{d^{|m|}P_l}{dx^{|m|}}$ an associated Legendre function, and $\epsilon = (-1)^m$ for $m \geq 0$ and $\epsilon = 1$ for $m < 0$. The $3p_z$ orbital is the case $n = 3$, $l = 1$, and $m = 0$. For the following problems, you should do them for this orbital, but feel free to try it for other orbitals or even the general one.

a) [10] Find the probability of finding the electron with ρ between a and $2a$.

b) [5] While the probability of finding the electron at any given radius is zero, find an integral that measures the relative probability of finding the electron at radius ρ.

c) [10] Evaluate your expression from part b. Find the radius at which it is most probable to find the electron.

Problem 2 (20) Consider the Klein bottle S, parameterized by

$$\begin{aligned}
x &= \cos u(\cos(\frac{u}{2})(\sqrt{2}+\cos v) + \sin(\frac{u}{2})\sin v\cos v) \\
y &= \sin u(\cos(\frac{u}{2})(\sqrt{2}+\cos v) + \sin(\frac{u}{2})\sin v\cos v) \\
z &= -\sin(\frac{u}{2})(\sqrt{2}+\cos v) + \cos(\frac{u}{2})\sin v\cos v
\end{aligned}$$

where $0 \leq u \leq 2\pi$ and $0 \leq v \leq 2\pi$. The Klein bottle is not orientable, so there is no continuous choice of normal vector and thus our definition of a flux integral is no longer well defined. But if we take the absolute value of $\vec{\mathbf{F}} \cdot \hat{\mathbf{n}}$ then we can integrate a vector field over this surface.

Let $\vec{\mathbf{F}}(x,y,z) = (0,0,1)$. Compute $\int\int_S |\vec{\mathbf{F}} \cdot \hat{\mathbf{n}}| dS$.

Problem 3 (68) *(A first course in complex variables, adapted from Spivak)*

If $f\colon \mathbb{C} \to \mathbb{C}$, define f to be *differentiable* at $z_0 \in \mathbb{C}$ if the limit

$$f'(z_0) = \lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}$$

exists. If f is differentiable at every point z in an open set A and f' is continuous on A, then f is called analytic on A.

a) [8] Show that $f(z) = z$ is analytic and $f(z) = \bar{z}$ is not (where bar denotes complex conjugation). Show that the sum, product, and quotient of analytic functions are analytic.

b) [10] If $f = u + iv$ is analytic on A, show that u and v satisfy the *Cauchy-Riemann equations:*

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

[Hint: Use the fact that $\lim_{z \to z_0}[f(z) - f(z_0)]/[z - z_0]$ must be the same for $z = z_0 + (x + i \cdot 0)$ and $z = z_0 + (0 + i \cdot y)$ with $(x, y) \to (0, 0)$. (Note that the converse is also true, if u and v are differentiable, though this is harder to prove).]

c) [8] Let $T\colon \mathbb{C} \to \mathbb{C}$ be a linear transformation (where $\mathbb{C}$ is considered as a vector space over $\mathbb{R}$). If the matrix of T with respect to the basis $(1, i)$ is $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ show that T is multiplication by a complex number if and only if $a = d$ and $b = -c$. Part b shows that an analytic function $f\colon \mathbb{C} \to \mathbb{C}$, considered as a function $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, has a derivative $Df(z_0)$ which is multiplication by a complex number. What complex number is this?

d) [10] Define

$$\begin{aligned} d(\omega + i\eta) &= d\omega + i\, d\eta, \\ \int_C \omega + i\eta &= \int_C \omega + i \int_C \eta, \\ (\omega + i\eta) \wedge (\theta + i\lambda) &= \omega \wedge \theta - \eta \wedge \lambda + i(\eta \wedge \theta + \omega \wedge \lambda), \\ dz &= dx + i\, dy \end{aligned}$$

Show that $d(f \cdot dz) = 0$ if and only if f satisfies the Cauchy Riemann equations.

e) [10] Prove the *Cauchy Integral Theorem:* If f is analytic on A, then $\int_C f dz = 0$ for every closed curve C such that C is the boundary of some region $R \subset A$.

f) [10] Show that if $g(z) = 1/z$ then $g \cdot dz = i\, d\theta + dh$ for some function $h\colon \mathbb{C} - 0 \to \mathbb{R}$. Conclude that if $C(R, n)$ is a curve that winds counterclockwise around a circle of radius R n times, then $\int_{C(R,n)} \frac{1}{z} dz = 2\pi i n$.

g) [12] If f is analytic on $\{z : |z| < 1\}$, use the fact that $g(z) = f(z)/z$ is analytic in $\{z : 0 < |z| < 1\}$ to show that

$$\int_{C(R_1,n)} \frac{f(z)}{z} dz = \int_{C(R_2,n)} \frac{f(z)}{z} dz$$

if $0 < R_1, R_2 < 1$. Use part f to evaluate $\lim_{R \to 0} \int_{C(R,n)} \frac{f(z)}{z} dz$ and conclude:

Cauchy Integral Formula: If f is analytic on $\{z : |z| < 1\}$ and C is a closed curve in $\{z : 0 < |z| < 1\}$ with winding number n around 0, then

$$n \cdot f(0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{z} dz.$$