18.02  Problem Set 5
(Due Tuesday, October 18, 11:59:59 PM)

Part I  (40 points)

HAND IN ONLY THE UNDERLINED PROBLEMS
(The others are some suggested choices for more practice.)
EP = Edwards and Penny; SN = Supplementary Notes (most have solutions)

Gradient, directional derivatives
Reading: EP §§13.8
Exercises:
EP §13.8 2, 7, 16, 19, 21, 32, 46, 51, 60
SN §2D 1, 2abc, 3, 4

Lagrange multipliers
Reading: EP §13.9
Exercises:
EP §13.9 13, 22, 30, 43, 49, 62, 63
SN §2I 1ab, 2

Part II  (26+12EC points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you
must write up solutions independently.

Problem 1  (14)
Consider a function \( f: \mathbb{R}^3 \to \mathbb{R} \) that you want to maximize subject to the constraint
\( g(x, y, z) = 0 \). Let \( S = \{(x, y, z) : g(x, y, z) = 0\} \). Using Lagrange multipliers we can
maximize \( f \) on \( S \) by requiring \( \nabla f = \lambda \nabla g \) for some \( \lambda \) and solving for \( \lambda \) as well as
the point \( x \in \mathbb{R}^3 \). Locally around \( x \), the graph of \( g(x, y, z) \) is a surface, so we should
expect the same kinds of local behavior for \( f \) restricted to \( S \) as we have for maps
from \( \mathbb{R}^2 \) to \( \mathbb{R} \). Given such an arbitrary \( f \) and \( g \), determine the analogue of the second
derivative test for critical points of \( f \) when restricted to \( S \). [Hint: look at the proof
of Lagrange’s method in the book]

Problem 2  (12; 6, 4, 2)
a) Consider a function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \). I give you the following information about \( f \):

(i) \( f \) has no isolated local maxima (a maximum point \( x \) is isolated if there is some neighborhood \( U \) of \( x \) such that for all \( x' \in U \), \( f(x') < f(x) \)).

(ii) The isolated minima of \( f \) occur precisely at the points \((x, y)\) where \( x \) and \( y \) are both integers. The value of \( f \) at all of these points is 0.

(iii) At any point of distance \( \frac{1}{3} \) from such an integral point, the directional derivative in the inward pointing direction is positive.

Either draw sufficiently many level curves of \( f \) to show its behavior (ie you should include at least four minima and draw level curves spanning the range of values \( f \) takes on), or give a formula for a function \( f \) satisfying the above conditions.

b) On a separate plot, draw the gradient field of \( f \).

c) Is \( f \) globally bounded? Give an argument supporting your answer.

Problem 3  (12 points extra credit; 1, 9, 2)

a) What is the ring of integers in \( \mathbb{Q}[\sqrt{26}] \)?

b) What is the class group?

c) What is the fundamental unit?