18.02 Problem Set 5

(Due Tuesday, October 18, 11:59:59 PM)

Part I (40 points)

HAND IN ONLY THE UNDERLINED PROBLEMS
(The others are *some* suggested choices for more practice.)
EP = Edwards and Penny; SN = Supplementary Notes (most have solutions)

Gradient, directional derivatives
Reading: EP §§13.8
Exercises:
EP §13.8 2, 7, 16, 19, <u>21</u>, <u>32</u>, 46, <u>51</u>, 60
SN §2D 1, 2a<u>b</u>c, 3, 4

Lagrange multipliers
Reading: EP §13.9
Exercises:
EP §13.9 <u>13</u>, 22, 30, <u>43</u>, 49, <u>62</u>, 63
SN §2I 1a<u>b</u>, 2

Part II (26+12EC points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently.

Problem 1 (14)
Consider a function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ that you want to maximize subject to the constraint $g(x, y, z) = 0$. Let $S = \{(x, y, z) : g(x, y, z) = 0\}$. Using Lagrange multipliers we can maximize f on S by requiring $\nabla f = \lambda \nabla g$ for some λ and solving for λ as well as the point $\mathbf{x} \in \mathbb{R}^3$. Locally around $\mathbf{x}$, the graph of $g(x, y, z)$ is a surface, so we should expect the same kinds of local behavior for f restricted to S as we have for maps from $\mathbb{R}^2$ to $\mathbb{R}$. Given such an arbitrary f and g, determine the analogue of the second derivative test for critical points of f when restricted to S. [Hint: look at the proof of Lagrange's method in the book]

Problem 2 (12; 6, 4, 2)

a) Consider a function $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$. I give you the following information about f:

(i) f has no isolated local maxima (a maximum point $\mathbf{x}$ is isolated if there is some neighborhood U of $\mathbf{x}$ such that for all $\mathbf{x}' \in U$, $f(\mathbf{x}') < f(\mathbf{x})$).

(ii) The isolated minima of f occur precisely at the points (x, y) where x and y are both integers. The value of f at all of these points is 0.

(iii) At any point of distance $\frac{1}{3}$ from such an integral point, the directional derivative in the inward pointing direction is positive.

Either draw sufficiently many level curves of f to show its behavior (ie you should include at least four minima and draw level curves spanning the range of values f takes on), or give a formula for a function f satisfying the above conditions.

b) On a separate plot, draw the gradient field of f.

c) Is f globally bounded? Give an argument supporting your answer.

Problem 3 (12 points extra credit; 1, 9, 2)

a) What is the ring of integers in $\mathbb{Q}[\sqrt{26}]$?

b) What is the class group?

c) What is the fundamental unit?