18.02 Problem Set 3

(Due Tuesday, October 4, 11:59:59 PM)

Part I (72 points)

HAND IN ONLY THE UNDERLINED PROBLEMS
(The others are *some* suggested choices for more practice.)
EP = Edwards and Penny; SN = Supplementary Notes (most have solutions)

Partial derivatives, differentiability, total derivative
Reading: EP §§13.2, 13.3, 13.4, end of 13.7
Exercises:
EP §13.2 <u>38</u>, 53, 54, 55, 56, 57, 58
EP §13.3 <u>41</u>, <u>51</u>, 53
EP §13.4 15, 16, 24, <u>41</u>, <u>56</u>, 58, <u>68</u>, 73, 74 (73 and 74 required for axiak)

Tangent Planes, linear approximation
Reading: EP §§13.6, 13.8 SN §TA
Exercises:
EP §13.4 39, <u>66</u>
EP §13.8 <u>31</u>, 34

Min-max problems, compact sets, least squares
Reading: EP §13.5, SN §LS
Exercises:
EP §13.5 11, <u>30</u>, 38, <u>49</u>, <u>59</u>, 61, 68
SN §2G 1<u>c</u>

Part II (28 points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently.

Problem 1 (18 + 10 extra credit) This problem is designed to give you a feel for how you might use this material in later classes here at MIT. This example, and the handout on modelling data using least squares, is taken from 12.410J (Observational Techniques in Optical Astronomy). The numbers involved are changed to ease computation, though you're still welcome to use Mathematica or Maple.

You're attempting to callibrate a spectrograph using an argon lamp. The spectrograph splits incoming light into its different wavelength components, which are then imaged on a CCD camera. But you don't know which wavelength corresponds to which column on the image. This is where the argon lamp comes in. Since argon emits radiation at a certain set of well defined wavelengths, you can match the wavelengths with the columns and use this callibration when you're taking images of stars or planets.

You have the following data points:

$$(7000\text{Å}, 640 \pm 2 \text{ pixels})$$
$$(7500\text{Å}, 530 \pm 4 \text{ pixels})$$
$$(8000\text{Å}, 410 \pm 8 \text{ pixels})$$
$$(9000\text{Å}, 200 \pm 3 \text{ pixels})$$
$$(9500\text{Å}, 130 \pm 12 \text{ pixels}).$$

Fit a quadratic of the form $y = a(x - x_0)^2 + b(x - x_0) + c$ to this data set, where y is the column (in pixels) corresponding to the wavelength x. Choose x_0 to be the average of the wavelengths of the given data points. The coefficients a, b and c are the variables that you are changing in order to optimize the fit. You may ignore the errors on the column measurements and assign all the points the same weight (though for extra credit read the material in the handout about weights and do the fitting with the given standard deviations instead of weighting all points the same. Note that ± 2 means that $\sigma = 2$ is the standard deviation).

Problem 2 (10; 4,2,4)
Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be given by

$$f(x, y, z) = (x^2 - yz, xy + xz + y).$$

a) Find an expression for $\mathrm{D}\, f(x, y, z)$.
b) Give the matrix representing $\mathrm{D}\, f(1, 2, 4)$.
c) In what directions can you move from $(1, 2, 4)$ and produce no first order change in the value of f?

Problem 3 (12 extra credit; 4,4,4) (adapted from Spivak, Calculus on Manifolds)
We've seen in class that if $f\colon \mathbb{R}^m \longrightarrow \mathbb{R}^n$ is differentiable, then all of it's partial derivatives exist. However, the converse is not true: there are examples of f such that all partials exist but f is not differentiable.
a) Let $A = \{(x, y) \in \mathbb{R}^2 | x > 0 \text{ and } 0 < y < x^2\}$. Define $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ by $f(\mathbf{x}) = 0$ if $\mathbf{x}$ is not in A and $f(\mathbf{x}) = 1$ otherwise. Prove that all directional derivatives of f exist at 0, even though f is not even continuous at 0.

If one requires in addition that all partials are continuous, then one can conclude

that f is differentiable. But not all differentiable functions have continuous partials:
b) Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f is differentiable at 0 but f' is not continuous at 0.
c) Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} (x^2 + y^2) \sin \frac{1}{\sqrt{x^2+y^2}} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f is differentiable at $(0, 0)$ but the partials are not continuous at $(0, 0)$.