18.02 Problem Set 1

(Due Tuesday, September 13, 11:59:59 PM)

Part I (56 points)

HAND IN ONLY THE UNDERLINED PROBLEMS
(The others are *some* suggested choices for more practice.)
EP = Edwards and Penny; SN = Supplementary Notes (most have solutions)

Vectors, coordinate systems, maps
Reading: EP §§12.1, 12.2, 10.2, 12.8
Exercises:
EP §12.1 (p. 777) 9, 10, 17, 19, 29, 47, $\underline{51}$. 54
SN §1A 5, $\underline{7}$, 9, 10, 11, $\underline{12}$
EP §12.8 (p. 843) 1, 10, 15, 33, 34, 36, $\underline{55}$

Linear maps, matrices, inverse matrices
Reading: SN §M
Exercises:
SN §1F 3, 4, 5a$\underline{b}$, 7, 8$\underline{a}$, $\underline{9}$
SN §1G $\underline{5}$, $\underline{8}$, 10

Determinants, dot product, cross product
Reading: EP §§12.2, 12.3, SN §D Exercises:
SN §1B 1, 2, 5b, 11, $\underline{12}$, $\underline{13}$, 14
SN §1C 1, $\underline{2}$, 3, 5$\underline{a}$, 6, 7
SN §1D 1, 2, 3, 4, $\underline{5}$, 7

Part II (44 points)

Directions: Try each problem alone for 20 minutes. If you collaborate later, you must write up solutions independently.

Problem 1 (6)
Describe the region of space inside a torus of inner radius of 3 and outer radius of 5 in the coordinate system of your choice.

Problem 2 (10; 5, 5)
Try to do the following two proofs in as much generalization as you can. If you need to restrict your attention to vectors in $\mathbb{R}^3$ though, you may do so. a) Prove the **Cauchy-Schwartz Inequality**:

$$|\mathbf{u} \cdot \mathbf{v}| \leq |\mathbf{u}||\mathbf{v}|.$$

b) Prove the **Triangle Inequality**:

$$|\mathbf{u} + \mathbf{v}| \leq |\mathbf{u}| + |\mathbf{v}|.$$

Problem 3 (10; 2, 2, 2, 2, 2)
The eight vertices of a cube centered at $(0, 0, 0)$ of side length 2 are at $(\pm 1, \pm 1, \pm 1)$.
a) Find the four vertices of the cube, including $(1, 1, 1)$ that form a regular tetrahedron.
b) A methane molecule consists of a hydrogen atom at each of the vertices of a regular tetrahedron and a carbon atom at the center. Find the "bond angle," i.e. the angle made by the vectors from the carbon atom to two hydrogen atoms.
c) Use the dot product to find the angle between two adjacent edges of the tetrahedron, and the angle between two opposite edges.
d) Find the area of a face of the tetrahedron using vectors.
e) Find the volume of the tetrahedron using vectors.

Problem 4 (10; 4, 6)
a) Let $V = \mathbb{R}^n$. We say that a function $f \colon V \times \cdots \times V \longrightarrow \mathbb{R}$ is *multilinear* if $f(v_1, \ldots, av_i + bv_i', \ldots, v_k) = af(v_1, \ldots, v_i, \ldots, v_k) + bf(v_1, \ldots, v_i', \ldots, v_k)$. One can think of the determinant map as a function of the rows of the matrix, in which case it maps from n copies of $\mathbb{R}^n$ to $\mathbb{R}$. Show that this map is multilinear.
b) We say that f is alternating if $f(v_1, \ldots, v, \ldots, v, \ldots, v_k) = 0$. Prove that the determinant map is alternating.

It turns out that these two properties, together with the property that $\det(e_1, \ldots, e_n) = 1$ uniquely characterize the determinant. Also note that the alternating condition implies that $f(v_1, \ldots, v, \ldots, w, \ldots, v_k) = -f(v_1, \ldots, w, \ldots, v, \ldots, v_k)$. The reverse implication holds, even replacing $\mathbb{R}$ by an arbitrary field k, as long as the characteristic of k is not 2. Can you prove these last statements?

Problem 5 (8)
Use Gaussian elimination to find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 0 \\ 4 & 10 & 6 \\ 2 & 4 & 2 \end{pmatrix}$$