### 18.02 ESG Final Exam Solutions Spring 2005

Write your name in the top right corner of this page. Work in the space provided or on the backs of pages. You are allowed five pages of notes and the use of at most two calculators, but you must show your work to get full credit and no other aids are allowed.

There are 255 points available. Full credit is 250 .

1. [20 points]

Consider the system of equations

$$
\begin{aligned}
x_{1}+5 x_{2}+3 x_{3} & =1 \\
x_{1}+7 x_{2}+7 x_{3} & =7 \\
3 x_{1}+19 x_{2}+17 x_{3} & =-2
\end{aligned}
$$

(a) [3] Rewrite this system in the matrix form $A \mathbf{x}=\mathbf{b}$.

Solution:

$$
\left(\begin{array}{ccc}
1 & 5 & 3 \\
1 & 7 & 7 \\
3 & 19 & 17
\end{array}\right)\left(\begin{array}{l}
x_{1} \\
x_{2} \\
x_{3}
\end{array}\right)=\left(\begin{array}{r}
1 \\
7 \\
-2
\end{array}\right)
$$

(b) [5] Is $A$ invertible? Justify your answer.

Solution:
No.
$\operatorname{det}(A)=1(7 \cdot 17-7 \cdot 19)+5(7 \cdot 3-1 \cdot 17)+3(1 \cdot 19-7 \cdot 3)=0$.
(c) [12] Give all solutions to the system. Make sure to include your steps.
Solution:
Using Gaussian elimination,

$$
\left.\begin{array}{l}
\left(\begin{array}{ccc|c}
1 & 5 & 3 & 1 \\
1 & 7 & 7 & 7 \\
3 & 19 & 17 & -2
\end{array}\right) \rightarrow\left(\begin{array}{ccc|c}
1 & 5 & 3 & 1 \\
0 & 2 & 4 & 6 \\
3 & 19 & 17 & -2
\end{array}\right) \rightarrow \\
\left(\begin{array}{lll|}
1 & 5 & 3 \\
0 & 2 & 4 \\
0 & 4 & 8
\end{array}\right. \\
\hline
\end{array}\right) \rightarrow\left(\begin{array}{lll|c}
1 & 5 & 3 \\
0 & 1 & 2 & 1 \\
0 & 4 & 8 & -5
\end{array}\right) \rightarrow+\left(\begin{array}{lll|c}
1 & 5 & 3 & 1 \\
0 & 1 & 2 & 3 \\
0 & 0 & 0 & -17
\end{array}\right) \rightarrow+
$$

The last equation says that $0=17$, so there are no solutions.
2. [10 points]

Consider the map $f: \mathbb{R}^{5} \longrightarrow \mathbb{R}^{2}$ given by the formula

$$
f(x, y, z, w, u)=(5 x+2 y-3 z+u, u-z+4 w) .
$$

What is the dimension of the kernel of $f$ ?

## Solution:

$f$ is clearly onto. So the dimension of the kernel is $5-2=3$ by the rank-nullity theorem.
3. [15 points]

Let $\mathbf{v}=(7,6,5)$ and $\mathbf{w}=(3,2,-1)$. Express $\mathbf{v}$ as the sum of two perpendicular vectors, one of which points in the direction of $\mathbf{w}$. [Hint: project $\mathbf{v}$ onto $\mathbf{w}$

## Solution:

The projection of $\mathbf{v}$ onto $\mathbf{w}$ is $\frac{\mathbf{v} \cdot \mathbf{w}}{|\mathbf{w}|^{2}} w$ which is $\frac{28}{14}(3,2,-1)=(6,4,-2)$. Subtracting this from $v$ gives $(1,2,7)$, so $(6,4,-2)$ and $(1,2,7)$ satisfy the given requirements.
4.
[25 points]
Find all critical points of the function $f(x, y, z)=x y^{3}-z$ when restricted to the surface $x y+y z=-3$

## Solution:

We use Lagrange multipliers. The equations are

$$
\begin{array}{r}
y^{3}=\lambda y \\
3 x y^{2}=\lambda(x+z) \\
-1=\lambda y
\end{array}
$$

The first and third equations imply that $y^{3}=-1$ so $y=-1$ and thus $\lambda=1$. Thus $3 x=x+z$ so $z=2 x$ and therefore $x y+y(2 x)=-3$ implies $x=1$ and thus $z=2$. So $(1,-1,2)$ is the only critical point.
5. [20 points]

Consider the function $f: \mathbb{R}^{3} \longrightarrow \mathbb{R}^{2}$ defined by

$$
f(x, y, z)=\left(x^{2} y-2 x z+1,5 z^{3}+4 x y^{2}\right)
$$

Approximate $f$ by a linear function near the point (1, 1,0 ).

## Solution:

$f$ is approximated by $f(1,1,0)+D f(1,1,0)(x-1, y-1, z) . f(1,1,0)=$ $(2,4)$ and the matrix of $\operatorname{Df}(1,1,0)$ is

$$
\left(\begin{array}{ccc}
2 x y-2 z & x^{2} & -2 x \\
4 y^{2} & 8 x y & 15 z^{2}
\end{array}\right)=\left(\begin{array}{ccc}
2 & 1 & -2 \\
4 & 8 & 0
\end{array}\right)
$$

so the linear approximation to $f$ is

$$
(x, y, z) \mapsto(2+2(x-1)+(y-1)-2(z), 4+4(x-1)+8(y-1))=(2 x+y-2 z-1,4 x+8 y-8)
$$

6. [25 points]

Let $f(x, y)=3 x y-x^{3}-y^{3}$. Find and classify all critical points of $f$.

## Solution:

We set all partials equal to zero, obtaining

$$
\begin{aligned}
& 3 y-3 x^{2}=0 \\
& 3 x-3 y^{2}=0
\end{aligned}
$$

and thus $y=y^{4}$ and $x=y^{2}$ so $x=0$ and $y=0$ or $x=1$ and $y=1$. The second partials are

$$
\begin{aligned}
& \frac{\partial^{2} f}{\partial x^{2}}=-6 x \\
& \frac{\partial^{2} f}{\partial x y}=3 \\
& \frac{\partial^{2} f}{\partial y^{2}}=-6 y
\end{aligned}
$$

So at $(0,0)$, we have $(0)(0)-(3)^{2}=-9<0$ and thus $(0,0)$ is a saddle. At $(1,1)$, we have $(-6)(-6)-(3)^{2}=27>0$ and $-6<0$ so $(1,1)$ is a maximum.
7. [20 points]

Find the area of the ellipse $(4 x-y)^{2}+(x-3 y)^{2}<1$ using an appropriate change of coordinates.
Solution:
Set $u=4 x-y$ and $v=x-3 y$. Then the Jacobian is the inverse of the absolute value of the determinant of $\left(\begin{array}{cc}4 & -1 \\ 1 & -3\end{array}\right)$, which is $\frac{1}{11}$. The integral of the function 1 over our ellipse thus transforms to the integral of $\frac{1}{11}$ over a circle of radius 1 in the $u v$-plane, which is $\frac{\pi}{11}$.
8. [25 points]

Let $\mathbf{F}(x, y, z)=\left(a x^{2} y+z^{2}, x^{3}+4 y^{3} z, b x z+y^{4}\right)$.
(a) [6] For what values of $a$ and $b$ will $\mathbf{F}$ be conservative?

Solution:
Take the curl of $\mathbf{F}$ and solve for $a$ and $b$ that make it zero.

$$
\nabla \times \mathbf{F}=\left(4 y^{3}-4 y^{3}, 2 z-b z, 3 x^{2}-a z^{2}\right)
$$

so $b=2$ and $a=3$.
(b) [12] Using these values of $a$ and $b$, find a function $f(x, y, z)$ such that $\mathbf{F}=\nabla f$.

## Solution:

Integrating the $x$ component with respect to $x$ gives $f(x, y, z)=$ $x^{3} y+x z^{2}+g(y, z)$, and examining the $y$ and $z$ components shows that $g(y, z)=y^{4} z$. So $f(x, y, z)=x^{3} y+x z^{2}+y^{4} z$.
(c) [7] Again using these values of $a$ and $b$, give a defining equation for a surface $S$ with the property that

$$
\int_{P}^{Q} \mathbf{F} \cdot \mathbf{T} d s=0
$$

for any two points $P$ and $Q$ lying on $S$ and any path between them.

## Solution:

By the fundamental theorem of line integrals,

$$
\int_{P}^{Q} \mathbf{F} \cdot \mathbf{T} d s=f(Q)-f(P)
$$

This will always be zero if $f$ is constant on $S$. So in particular, the surface defined by the equation $x^{3} y+x z^{2}+y^{4} z=4$ will work. (And don't be nitpicky and give me the empty set for $S$.)
9. [15 points]

Let $C$ be the spiral given in polar coordinates by the equation $r=\theta$, $0 \leq \theta \leq 2 \pi$, traced out starting from the origin. Let $\mathbf{F}(x, y)=\left(x^{2}, y\right)$. Reduce the following line integral to a single variable integral, but do not evaluate the resulting single variable integral:

$$
\int_{C} \mathbf{F} \cdot d \mathbf{r} .
$$

Solution: Parameterizing with respect to theta,

$$
\int_{C} \mathbf{F} \cdot d \mathbf{r}=\int_{0}^{2 \pi}\left(\theta^{2} \cos ^{2} \theta, \theta \sin \theta\right) \cdot(-\sin \theta, \cos \theta) d \theta=\int_{0}^{2 \pi} \theta \sin \theta \cos \theta-\theta^{2} \sin \theta \cos ^{2} \theta d \theta
$$

10. [55 points]

Consider the sphere $S$ of radius 1 centered at the point $(0,0,1)$
(a) [10] Parameterize $S$. [Hint: how would the parameterizations of a sphere centered at the origin and a sphere centered at $(0,0,1)$ differ?] There are two easy parameterizations:

$$
\begin{aligned}
& x=\cos \theta \sin \phi \\
& y=\sin \theta \sin \phi \\
& z=1+\cos \phi \\
& 0 \leq \theta \leq 2 \pi, 0 \leq \phi \leq \pi
\end{aligned}
$$

or

$$
\begin{aligned}
& x=2 \cos \theta \sin \phi \cos \phi \\
& y=2 \sin \theta \sin \phi \cos \phi \\
& z=2 \cos ^{2} \phi \\
& 0 \leq \theta \leq 2 \pi, 0 \leq \phi \leq \frac{\pi}{2}
\end{aligned}
$$

(b) [15] Let $f(x, y, z)=x^{2} z+y^{2} z-x^{2}-y^{2}$. Compute

$$
\iint_{S} f(x, y, z) d S
$$

[Can you tell what the scaling factor (ie $\left|\mathbf{r}_{u} \times \mathbf{r}_{v}\right|$ ) is at a glance?]

## Solution:

If we use the first parameterization, the scaling factor is just $\sin \phi$ as in a normal sphere. So

$$
\begin{aligned}
\iint_{S} f(x, y, z) d S & = \\
d \operatorname{dint} S\left(x^{2}+y^{2}\right)(z-1) d S & \\
& =\int_{0}^{2 \pi} \int_{0}^{\pi} \sin ^{2} \phi \cos \phi \sin \phi d \phi d \theta \\
& =(2 \pi)\left(\left.\frac{1}{4} \sin ^{4} \phi\right|_{0} ^{\pi}\right. \\
& =0 .
\end{aligned}
$$

(c) [15] Let $\mathbf{F}(x, y, z)=(x+3 y, 2 y-z, 4 z+x)$. Compute

$$
\oiiint_{S} \mathbf{F} \cdot \mathbf{n} d S .
$$

## Solution:

Using the divergence theorem, this is just the integral of the function $1+2+4=7$ over the interior of the sphere. So the answer is $\frac{28 \pi}{3}$.
(d) [15] Now consider just the upper half of $S$ (above the $z=1$ plane). Call this surface $S_{1}$ and equip it with the upward pointing normal. Let $\mathbf{F}=\left(z-y-1, x+z^{2}+z-2, x y^{2}+x z-4\right)$. Compute

$$
\iint_{S_{1}}(\boldsymbol{\nabla} \times \mathbf{F}) \cdot \mathbf{n} d S
$$

## Solution:

By Stokes' theorem, we can just compute a line integral. Setting $z=1$ in $F$ we find that we're integrating $\left(-y, x, x y^{2}+x-4\right)$ around a circle of radius 1. Using Green's theorem, this is equivalent to integrating 2 over the disc of radius 1 , so the answer is $2 \pi$.
11. [25 points] Consider the part of the surface $z=-r^{2}+3 r-2$ that lies above the $x y$-plane ( $r$ is the $r$ of cylindrical coordinates). Call this surface $S$. Let

$$
\mathbf{F}(x, y, z)=\left(x+3 y-\sin ^{4}\left(z^{5}\right), x^{2}-y^{2},-z-4\right)
$$

Evaluate

$$
\iint_{S} \mathbf{F} \cdot \mathbf{n} d S .
$$

[Hint: use the divergence theorem so that the surface integral that you actually compute is easier.]
Solution: The region is a doughnut shaped thing that is flat on the bottom and parabolic on top. $S$ is above the $x y$-plane between $r=1$ and $r=2$. Note that this region is symmetric about the $x$-axis, and the divergence of $\mathbf{F}$ is just $-2 y$, so the integral of the divergence over this region is zero. Thus we can set $z=0$ and $\hat{\mathbf{n}}=\hat{\mathbf{k}}$ and integrate from $r=1$ to $r=2$. So the answer is $(-4)(4 \pi-\pi)=-12 \pi$.

