18.782 Introduction to Arithmetic Geometry — **Fall 2020**

Problem Set #9 — **Due: 11/6/2020**

These problems are related to the material covered in Lectures 22-25. I have made every effort to proof-read them, but some errors may remain. The first person to spot each error will receive 1-5 points of extra credit.

The problem set is due by 11:59pm on 11/6/2020 and should be submitted electronically as a pdf-file e-mailed to zzyzhang@mit.edu and roed@mit.edu (please include "18.782" in the subject of the email). You can use the latex source for this problem set as a template for writing up your solutions; be sure to include your name in your solutions and to identify collaborators and any sources not listed in the syllabus.

Recall that we have defined a *curve* as a smooth projective variety of dimension one.

Problem 1. Divisor identities (50 points)

Let k be a perfect (but not necessarily algebraically closed) field. Let $\phi\colon C_1 \to C_2$ be a morphism of curves defined over k, and let $\phi^*\colon k(C_2) \to k(C_1)$ be the corresponding embedding of function fields. We also use ϕ^* to denote the pullback map $\operatorname{Div}_k C_2 \to \operatorname{Div}_k C_1$, and ϕ_* denotes the pushforward map $\operatorname{Div}_k C_1 \to \operatorname{Div}_k C_2$, as defined in Definition 19.19.

(a) Prove that $\phi_* \circ \phi^*$ acts as multiplication by $\deg \phi$ on $\operatorname{Div}_k C_2$.

(b) Show that $\phi^* \circ \phi_*$ is not multiplication by $\deg \phi$ on $\operatorname{Div}_k C_1$, in general, but that we do have

$$\deg(\phi^*\phi_* D) = (\deg \phi)(\deg D)$$

for all $D \in \operatorname{Div}_k C_1$.

(c) Prove that $\operatorname{ord}_P(\phi^* f) = e_\phi(P) \operatorname{ord}_{\phi(P)}(f)$ for all closed points P of C_1/k and functions f in $k(C_2)^\times$. Then use this to show $\phi^*(\operatorname{div} f) = \operatorname{div}(\phi^* f)$ for all $f \in k(C_2)^\times$.

(d) For any curve C/k and $f \in k(C)^\times$ and $D \in \operatorname{Div}_k(C)$ such that $\operatorname{div} f$ and $D = \sum n_P P$ have disjoint supports, define

$$f(D) = \prod f(P)^{n_P \deg P},$$

where P ranges over the closed points of C/k. Prove that $f(\phi_* D) = (\phi^* f)(D)$ for all $f \in k(C_2)^\times$ and $D \in \operatorname{Div}_k(C_1)$ where both sides are defined.

(e) Prove that the sequence

$$1 \to k^\times \to k(\mathbb{P}^1)^\times \xrightarrow{\operatorname{div}} \operatorname{Div}_k \mathbb{P}^1 \xrightarrow{\deg} \mathbb{Z} \to 0$$

is exact.

Problem 2. Weil reciprocity (50 points)

Let C/k be a curve over an algebraically closed field k. For $f \in k(C)^\times$ and $D \in \operatorname{Div}_k C$ such that $\operatorname{div} f$ and D have disjoint support, define $f(D)$ as in part (d) of Problem 1. Your goal in this problem is to prove the *Weil reciprocity law*, which states that for all $f, g \in k(C)^\times$ such that $\operatorname{div} f$ and $\operatorname{div} g$ have disjoint support,

$$f(\operatorname{div} g) = g(\operatorname{div} f).$$

(a) Prove that for $C = \mathbb{P}^1$ we have

$$\prod_P (-1)^{\operatorname{ord}_P(g)\operatorname{ord}_P(f)} \left(\frac{f^{\operatorname{ord}_P(g)}}{g^{\operatorname{ord}_P(f)}}\right)(P) = 1,$$

for all $f, g \in k(C)^\times$, whether or not $\operatorname{div} f$ and $\operatorname{div} g$ have disjoint support.

(b) Use (a) to prove the Weil reciprocity law in the case $C = \mathbb{P}^1$.

Associated to any morphism of curves $\phi\colon C_1 \to C_2$ we have defined three maps

$$\begin{array}{ll} \phi^*\colon k(C_2) \to k(C_1) & \quad \phi^*\colon \operatorname{Div}_k C_2 \to \operatorname{Div}_k C_1 \\ ? & \quad \phi_*\colon \operatorname{Div}_k C_1 \to \operatorname{Div}_k C_2 \end{array}$$

It is natural to ask whether there is a fourth map $\phi_*\colon k(C_1) \to k(C_2)$ that completes the table above, and indeed there is. It is defined by

$$f \mapsto (\phi^*)^{-1} N(f),$$

where $N\colon k(C_1) \to \phi^*(k(C_2))$ is the norm map associated to the extension $k(C_1)/\phi^*(k(C_2))$.[1] It is obviously not a morphism of fields (unless ϕ is an isomorphism), but it is a morphism of their multiplicative groups.

We then have an identities analogous to those proved in (c) and (d) of Problem 1:

$$\phi_*(\operatorname{div} f) = \operatorname{div}(\phi_* f), \tag{1}$$

for all $f \in k(C_1)^\times$, and

$$f(\phi^* D) = (\phi_* f)(D), \tag{2}$$

for all $f \in k(C_1)^\times$ and $D \in \operatorname{Div}_k C_2$ where both sides are defined.

We now turn to the general case of the Weil reciprocity law, where f and g are elements of $k(C)$ with disjoint support, where C is now any curve. Your objective is to prove Weil's reciprocity law in the general case using Weil reciprocity for $\mathbb{P}^1$, which you proved in (b).

As noted in lecture, any function in $k(C)$ defines a morphism to $\mathbb{P}^1$, so we may view both f and g as morphisms, and we then have associated pullback maps f^*, g^* and pushforward maps f_*, g_* that we can apply both to the divisor groups and function fields of C and $\mathbb{P}^1$. We also need the identity morphism $i\colon \mathbb{P}^1 \to \mathbb{P}^1$.

(c) Prove that $\operatorname{div} g = g^* \operatorname{div} i$ for any $g \in k(C)^\times$.

(d) Use (b) and (c), parts (c) and (d) of Problem 1, and identities (1) and (2) to prove the Weil reciprocity law in the general case.

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[1] Recall from Definition 7.1 that the norm map of a finite extension L/K sends an element of L with characteristic polynomial $F \in K[x]$ to $(-1)^{[L:K]}F(0)$ which is an element of K.

Problem 3. Survey

Complete the following survey by rating each problem on a scale of 1 to 10 according to how interesting you found the problem (1 = "mind-numbing," 10 = "mind-blowing"), and how difficult you found the problem (1 = "trivial," 10 = "brutal"). Also estimate the amount of time you spent on each problem.

	Interest	Difficulty	Time Spent
Problem 1			
Problem 2			

Feel free to record any additional comments you have on the problem sets or lectures; in particular, how you think they might be improved.